Reachability in Vector Addition Systems

Jérôme Leroux and Sylvain Schmitz

EJCIM 2020

Outline

vector addition systems (VAS)

- central model of computation

reachability problem

- hard conceptually and computationally
- decision via decomposition algorithm

this lecture

- complexity upper bounds

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Vector Addition Systems (with States)

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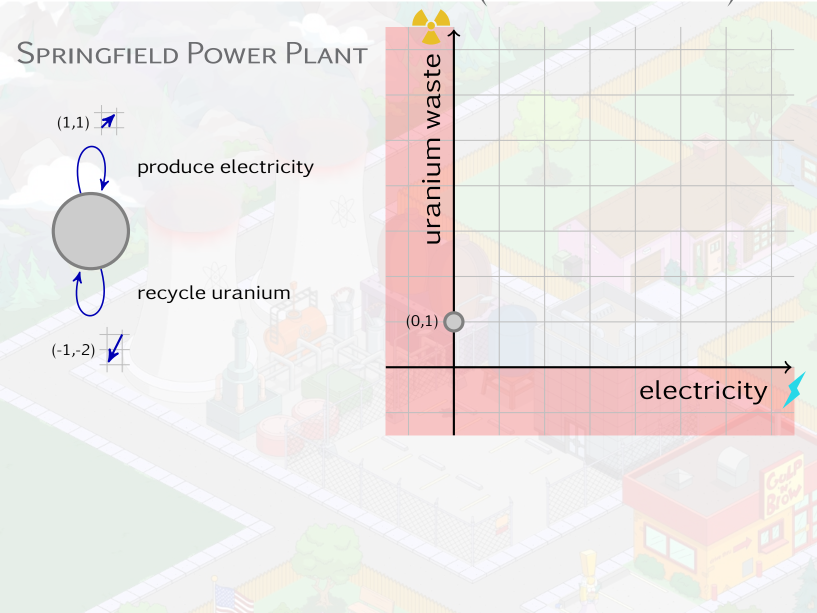
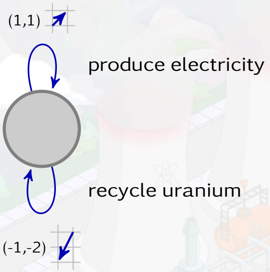
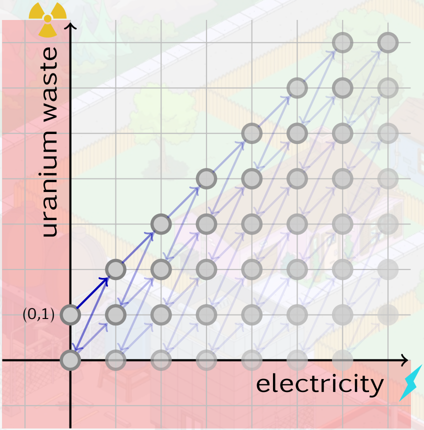


Vector Addition Systems (with States)

Springfield Power Plant

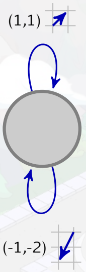


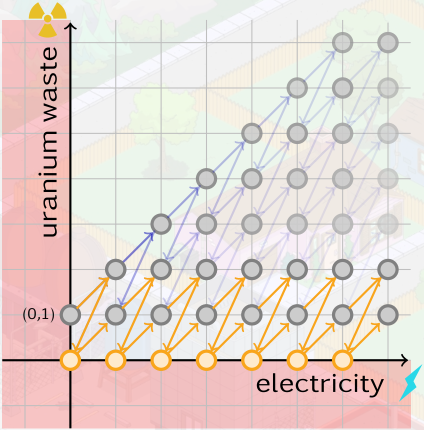


Can we produce unbounded electricity with no left-over uranium waste?

Vector Addition Systems (with States)

Springfield Power Plant





Can we produce unbounded electricity with no left-over uranium waste? Yes, $(\infty, 0)$ is reachable

Importance of the Problem

Modelling discrete resources

items, money, molecules, active threads, active data domain, ...

Central decision problem

Large number of problems interreducible with reachability in vector addition systems

- correctness of population protocols
- satisfiability of logics over data words
- provability of !-Horn linear logic
- ...

Current Upper Bounds

$$F_0(x) = x + 1$$

$$F_1(x) = \overbrace{F_0 \circ \cdots \circ F_0}^{x+1 \text{ times}}(x) = 2x + 1$$

$$F_2(x) = \overbrace{F_1 \circ \cdots \circ F_1}^{x+1 \text{ times}}(x) \approx 2^x$$

$$F_3(x) = \overbrace{F_2 \circ \cdots \circ F_2}^{x+1 \text{ times}}(x) \approx \text{tower}(x)$$

$$\vdots$$

$$F_\omega(x) = F_{x+1}(x) \qquad \approx \text{ackermann}(x)$$

Upper Bound Theorem ([Leroux & S. '19])
VAS Reachability is in $\mathbf{F}_\omega$, and in $\mathbf{F}_{d+4}$ in fixed dimension d

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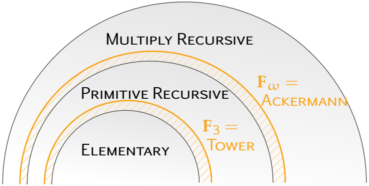

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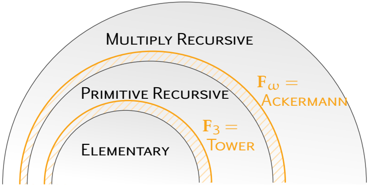

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Decomposition Algorithm

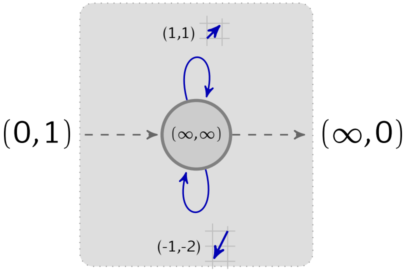
Ernst W. Mayr
[Mayr '81]

S. Rao Kosaraju
[Kosaraju '82]

Jean-Luc Lambert
[Lambert '92]

"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]



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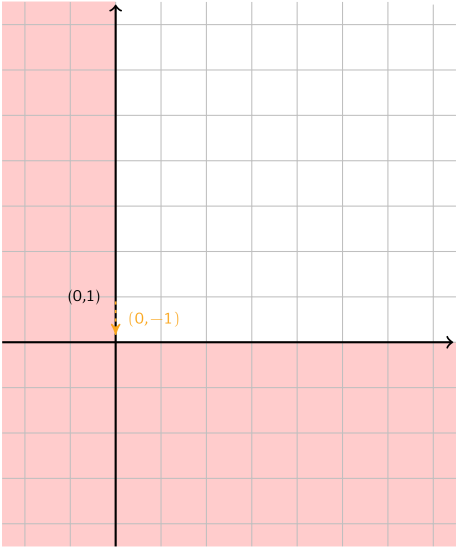

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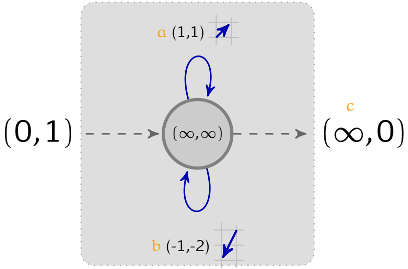


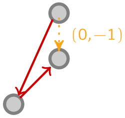
Characteristic System

$$0 + 1 \cdot a - 1 \cdot b = c$$
$$1 + 1 \cdot a - 2 \cdot b = 0$$

Solution for a, b

[1 · ↗, 1 · ↙]

Solution Path



"Simple Runs" (Θ Condition)

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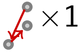

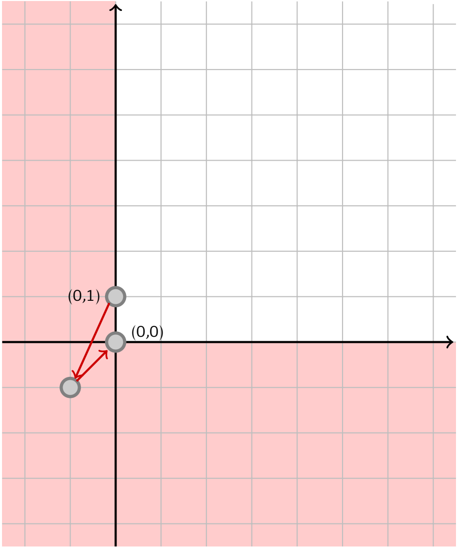

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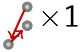

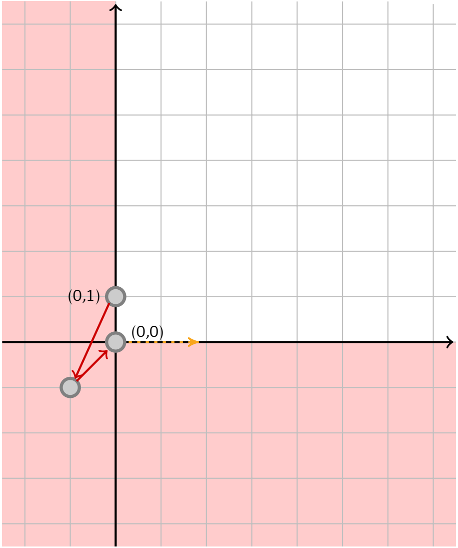

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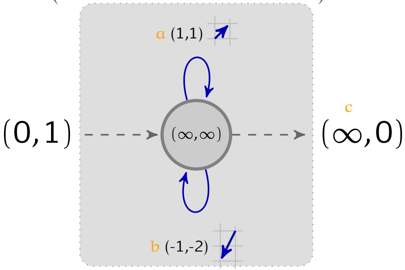


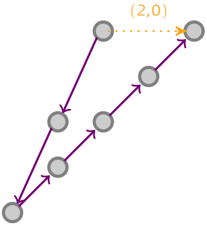
Homogeneous System

$$1 \cdot a - 1 \cdot b = c$$
$$1 \cdot a - 2 \cdot b = 0$$
$$a, b, c > 0$$

Solution for a, b

[4 · ↗, 2 · ↙]

Unbounded Path



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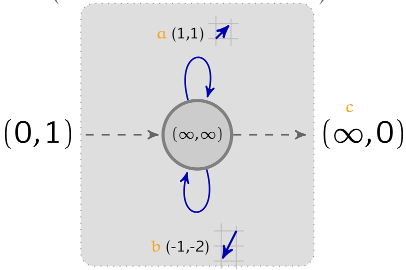
[Mayr'81, Kosaraju'82, Lambert'92]



Homogeneous System

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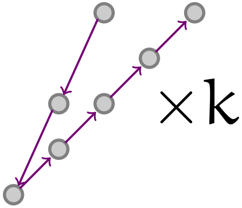
$$a, b, c > 0$$

Solution for a, b

$$[4k \cdot \nearrow, 2k \cdot \swarrow]$$

Unbounded Path



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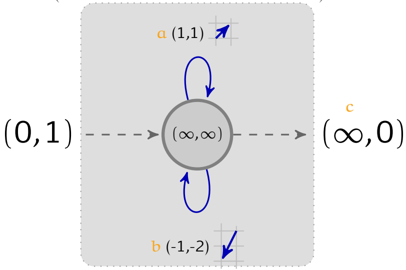

Characteristic System

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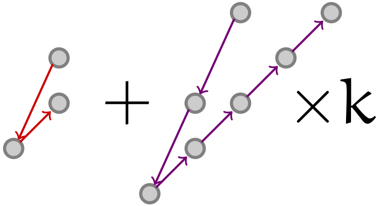
$$1 + 1 \cdot a - 2 \cdot b = 0$$

Solution for a, b

$[(1 + 4k) \cdot \nearrow, (1 + 2k) \cdot \swarrow]$

Path



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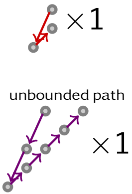

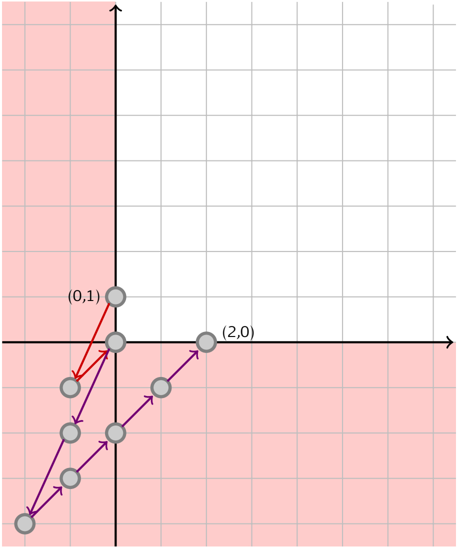

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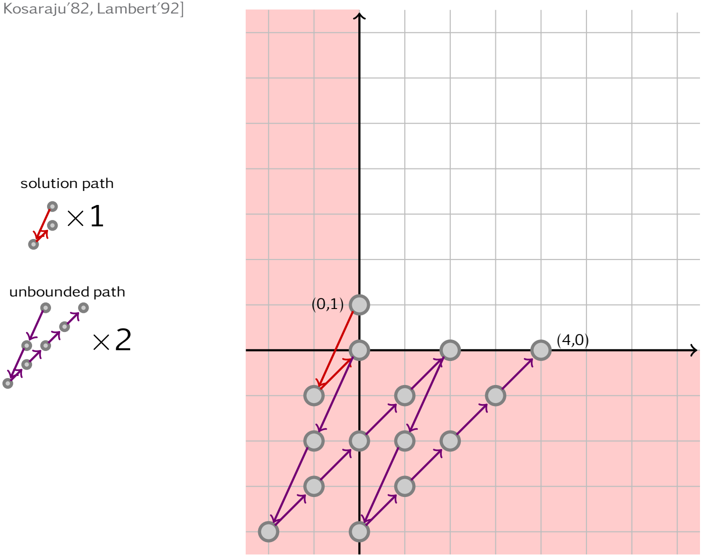

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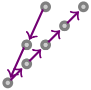
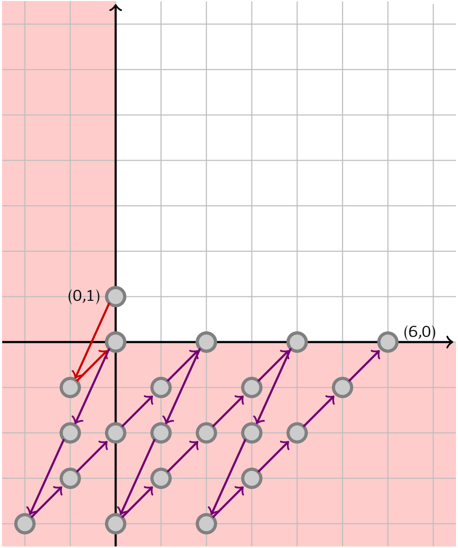
[Mayr'81, Kosaraju'82, Lambert'92]

solution path

unbounded path





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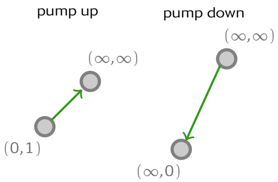
Pumpable Paths



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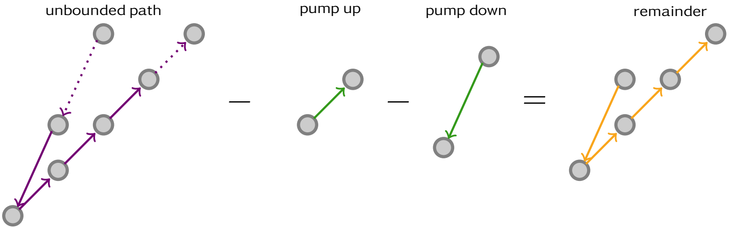
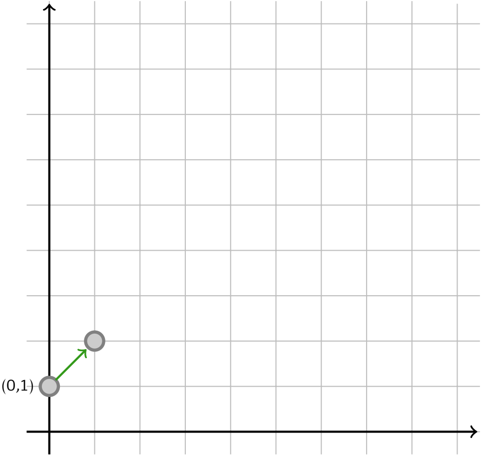


"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

pump up



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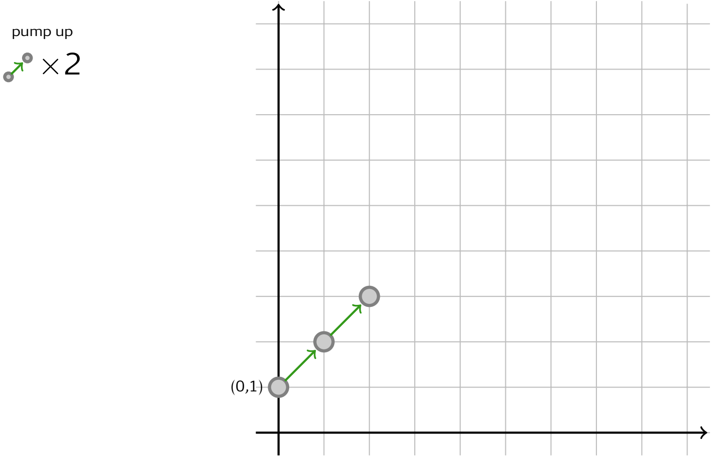
[Mayr'81, Kosaraju'82, Lambert'92]



"Simple Runs" (Θ Condition)

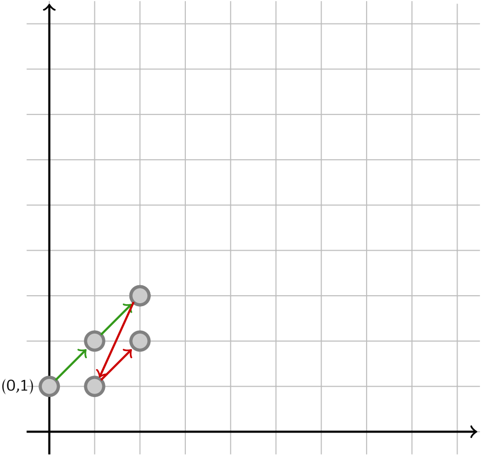
[Mayr'81, Kosaraju'82, Lambert'92]

pump up

$\times 2$

solution path

$\times 1$



"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

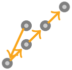
pump up

×2

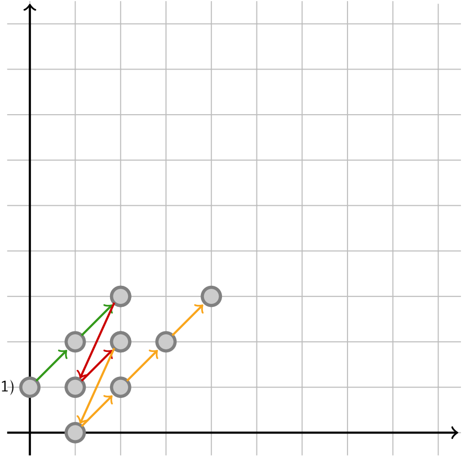
solution path

×1

remainder



×1



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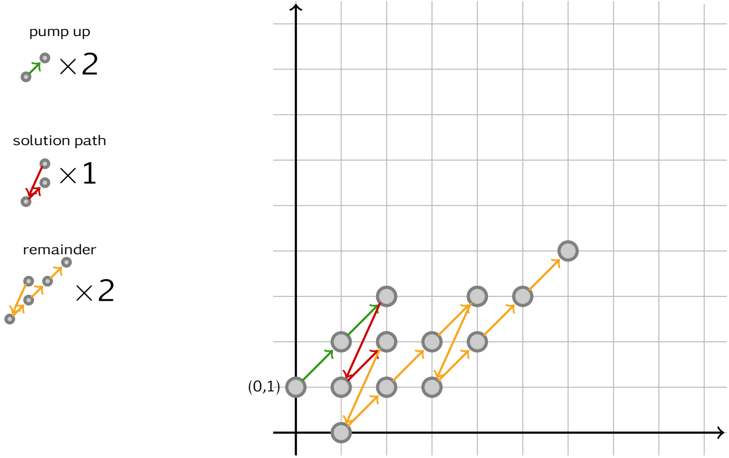

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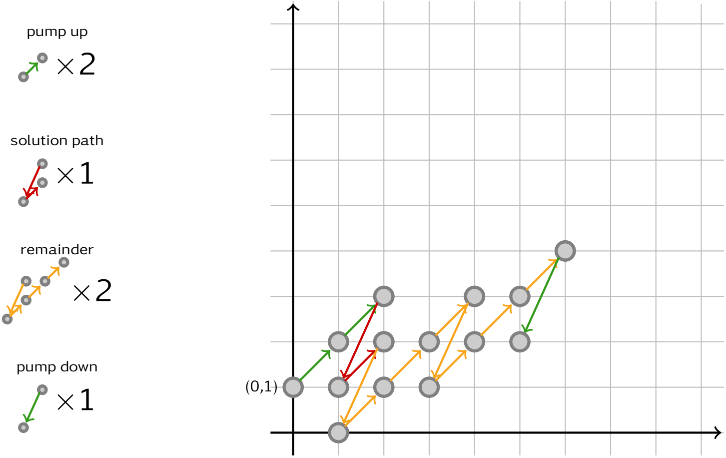


"Simple Runs" (Θ Condition)

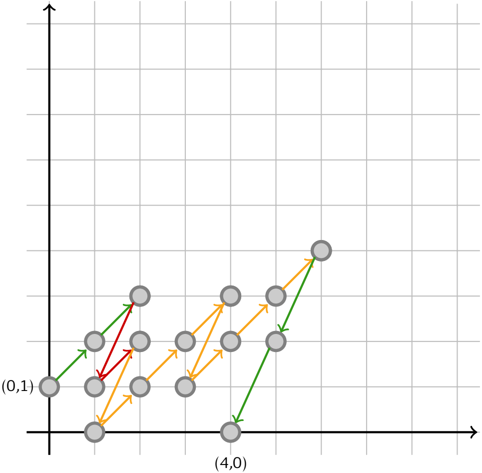
[Mayr'81, Kosaraju'82, Lambert'92]

pump up ×2

solution path ×1

remainder ×2

pump down ×2



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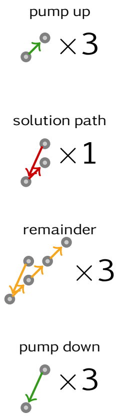
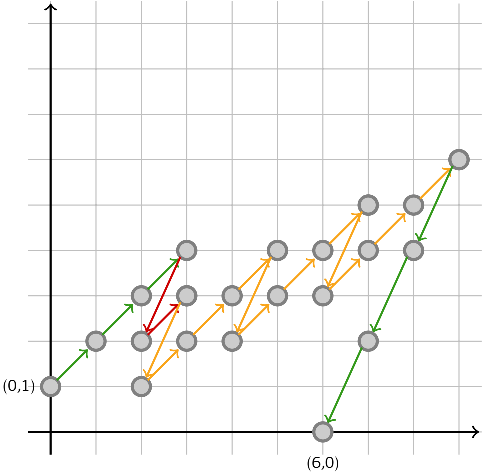
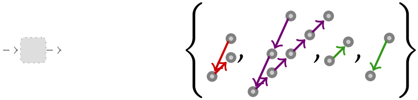




Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

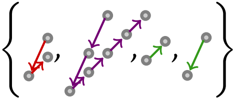
can we build a "simple run"?



Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

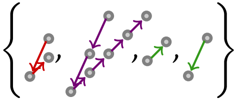
can we build a "simple run"? yes

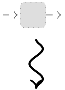


Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

can we build a "simple run"? no





☹

decompose

Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

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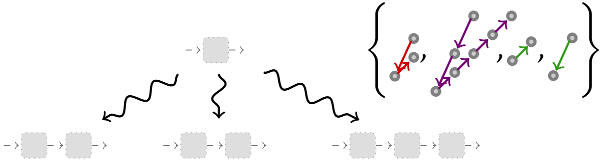

decompose

Decomposition Algorithm

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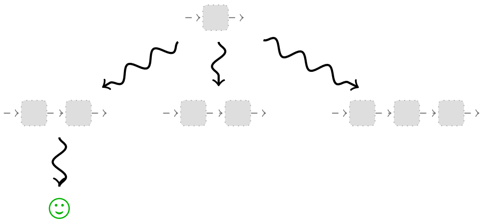

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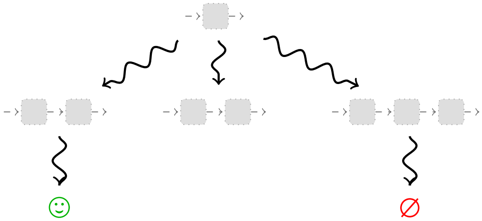

Decomposition Algorithm

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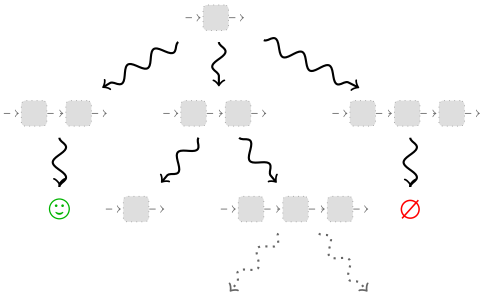
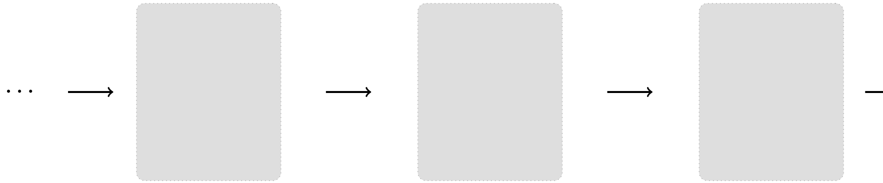


How to Decompose

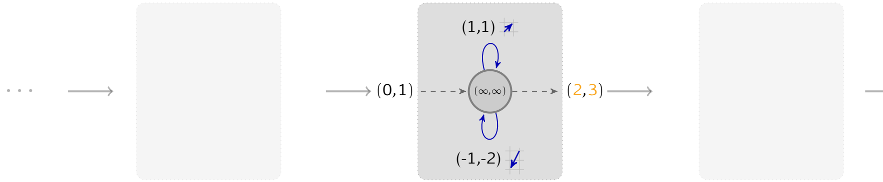
[Mayr'81, Kosaraju'82, Lambert'92]



How to Decompose

[Mayr'81, Kosaraju'82, Lambert'92]

No simple path:



How to Decompose

[Mayr'81, Kosaraju'82, Lambert'92]

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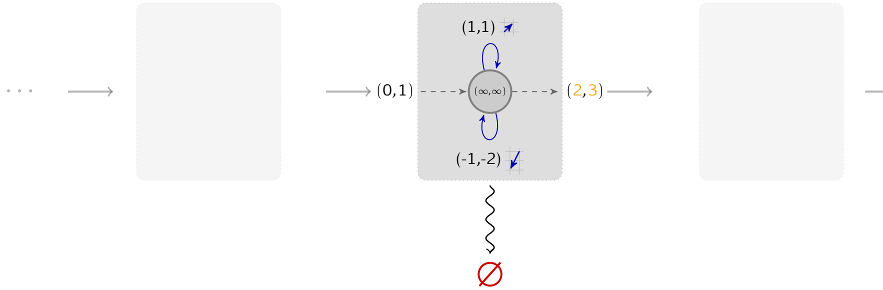
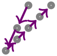
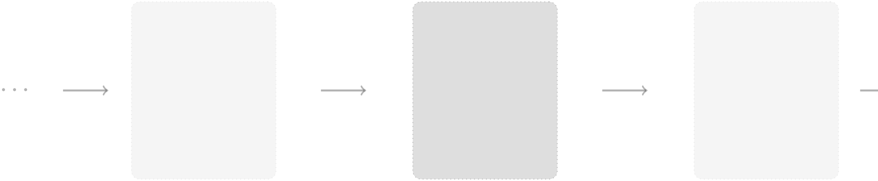


How to Decompose

[Mayr'81, Kosaraju'82, Lambert'92]

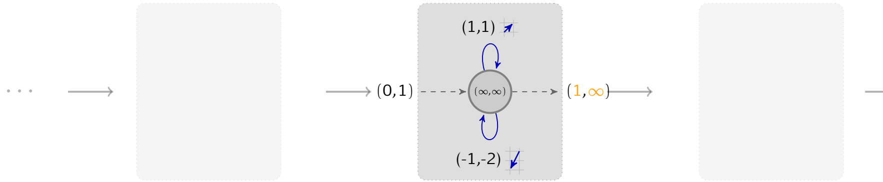
No unbounded path :



How to Decompose

[Mayr'81, Kosaraju'82, Lambert'92]

No unbounded path : Case of bounded '∞'



How to Decompose

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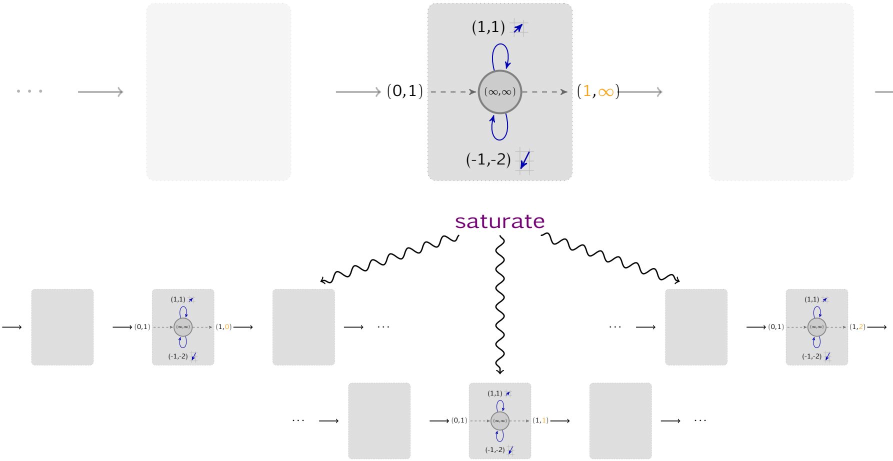
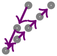
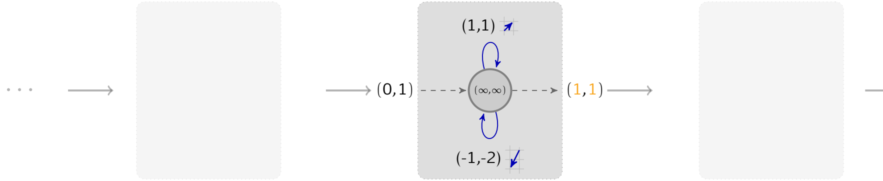
No unbounded path: Case of bounded '∞'



How to Decompose

[Mayr'81, Kosaraju'82, Lambert'92]

No unbounded path : Case of bounded transitions



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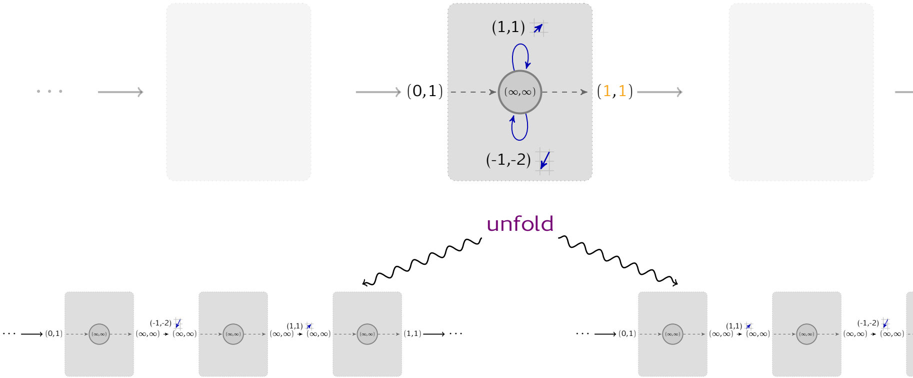

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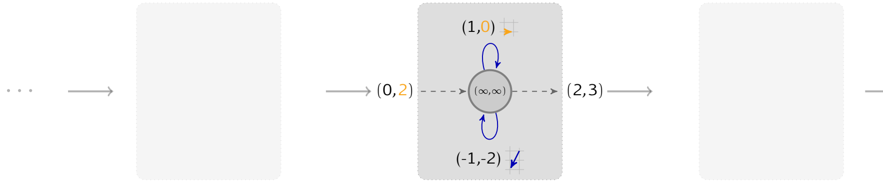
No pumping path or :



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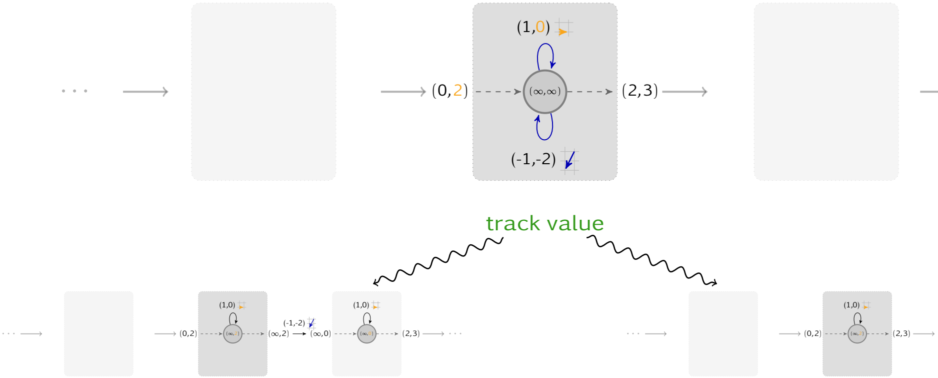


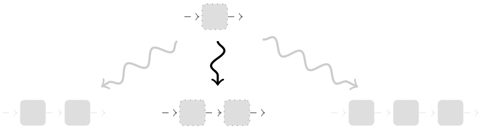
Termination

Ordinal Ranking Function

α_0

Termination

Ordinal Ranking Function



α_0

$\vee$

α_1

Termination

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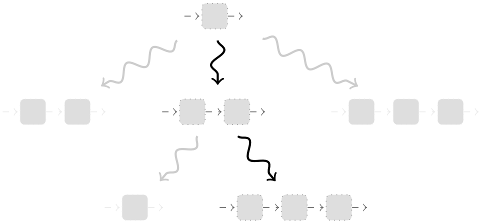

$$\alpha_0 > \alpha_1 > \alpha_2$$

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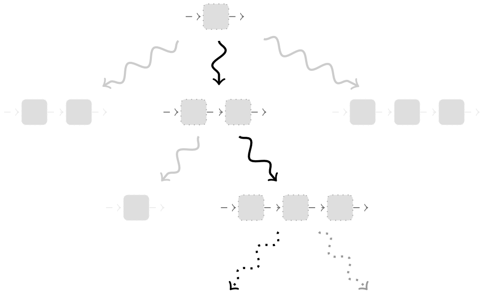

$$\alpha_0 > \alpha_1 > \alpha_2 > \cdots$$

New Ingredients

[Leroux & S. '19]

Technical Ingredients

[Leroux & S. '19]

1. new ranking function:

   order type ω^{d+1}

   ω^{ω^3} in [Leroux & S. '15]

   $\omega^{\omega} \cdot (d+1)$ in [S. '17]

2. refined analysis of pumpable paths:

   Rackoff-style analysis

   improves complexity from $\mathbf{F}_{2d+2}$ to $\mathbf{F}_{d+4}$

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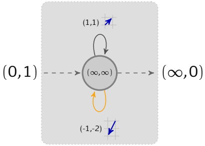
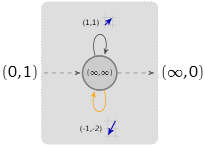
Rank of a Transition

For a transition t in 

$$\{\text{effects of cycles } C \mid t \in C\}$$

Rank of a Transition

For a transition t in 

$$\{m \cdot \nearrow + n \cdot \swarrow \mid m \geqslant 0, n > 0\}$$

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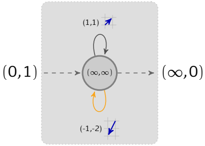
For a transition t in 

$$\mathrm{span}_{\mathbb{Q}}\left(\left\{m \cdot \nearrow + n \cdot \swarrow \mid m \geqslant 0, n > 0\right\}\right) = \mathbb{Q}^2$$

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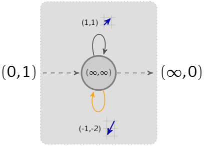
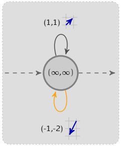
For a transition t in 

$$\dim\left(\mathrm{span}_{\mathbb{Q}}\left(\{m \cdot \nearrow + n \cdot \swarrow \mid m \geqslant 0, n > 0\}\right) = \mathbb{Q}^2\right) = 2$$

Rank of a Transition

For a transition t in (0,1)  (∞,0)

$$\dim\left(\operatorname{span}_{\mathbb{Q}}\left(\{m\cdot \nearrow + n\cdot \swarrow \mid m \geqslant 0, n > 0\}\right) = \mathbb{Q}^2\right) = 2$$

here, $\operatorname{rank}(t) = (1,0,0) \quad \in \mathbb{N}^{d+1}$

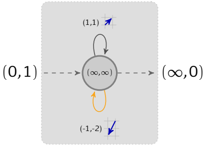
Definition

$$\operatorname{rank}(G) \stackrel{\text{def}}{=} \sum_{t\in G} \operatorname{rank}(t) \qquad \in \mathbb{N}^{d+1}$$

(added componentwise) (ordered lexicographically)

Rank of a VAS

For a transition t in



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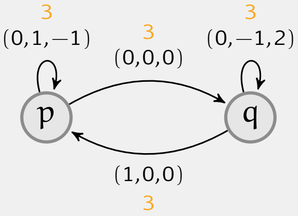
Rank of a VAS

Example



Example

Rank of a VAS

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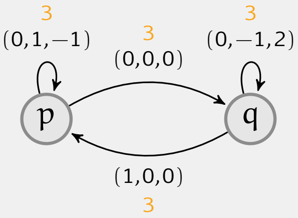

$\text{rank}(G) = (4,0,0,0)$

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Rank of a VAS

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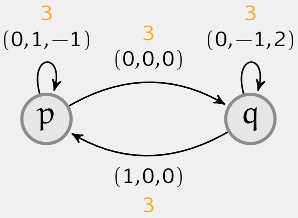

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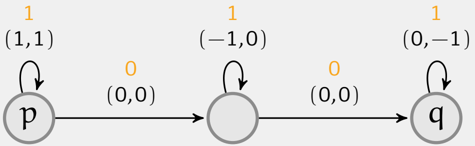

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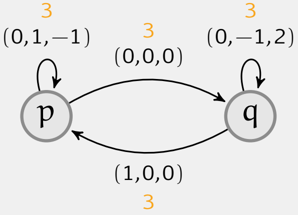

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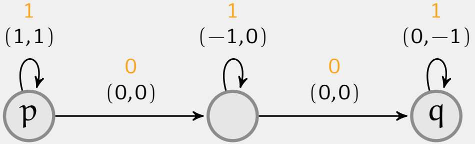
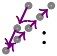
Example



$\mathrm{rank}(G) = (0,3,2)$

Decreasing Ranks

Recall decomposition steps:

- no : ∅
- no :
  - bounded '∞': saturate
  - bounded transitions: unfold
- no  or no : track value

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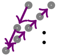
- no :
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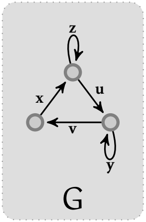
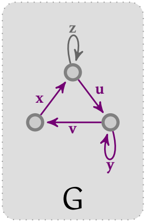
Decreasing Ranks when Unfolding



Decreasing Ranks when Unfolding

$T \setminus T'$: not in any homogeneous solution



T': in an homogeneous solution

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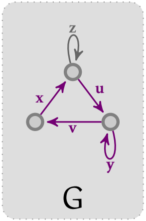

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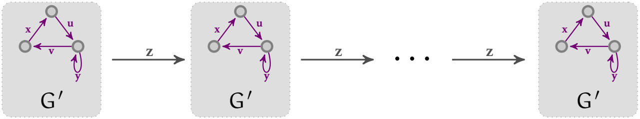

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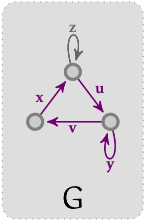

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Claim

If $T' \subsetneq T$, *then* $\mathrm{rank}(G') < \mathrm{rank}(G)$

- let $\mathbf{V}$, resp. $\mathbf{V}'$, be the vector space spanned by the cycles of T, resp. T'
- we want to show $\dim(\mathbf{V}') < \dim(\mathbf{V})$
- as $\mathbf{V}' \subseteq \mathbf{V}$, it suffices to show that $\mathbf{V}' = \mathbf{V}$ implies $T' = T$

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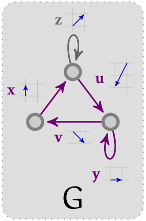

T': in an homogeneous solution

- assume $\mathbf{V}' = \mathbf{V}$.
- pick a cycle of G using every transition in T e.g., $\mathbf{x} + \mathbf{z} + \mathbf{u} + \mathbf{y} + \mathbf{v}$
- the effect of the cycle is $\Delta \in \mathbf{V}$
- as $\mathbf{V} = \mathbf{V}'$, there exists a rational linear combination of cycles of T' with effect Δ e.g., $\Delta = \frac{1}{2}(\mathbf{x} + \mathbf{u} + 4\mathbf{y} + \mathbf{v})$
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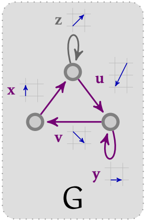

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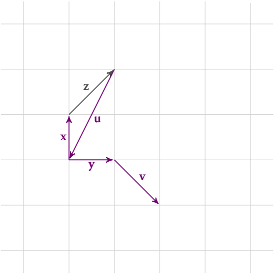

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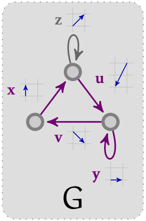

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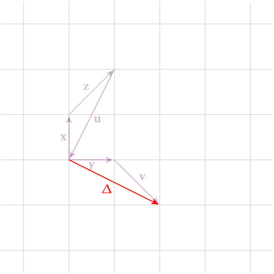

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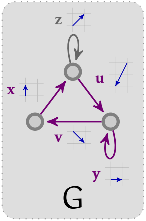

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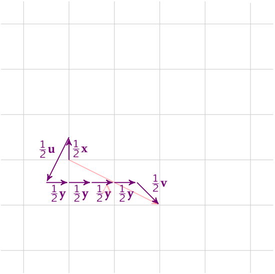

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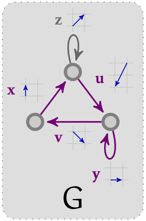

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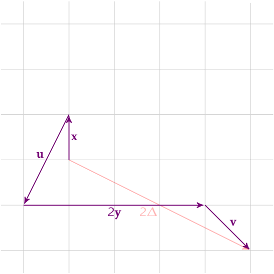

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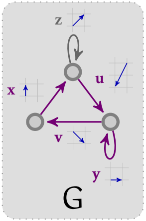

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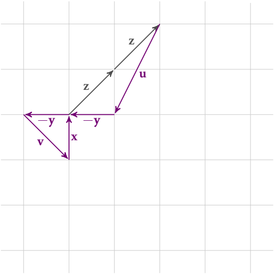

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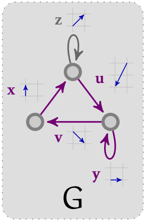
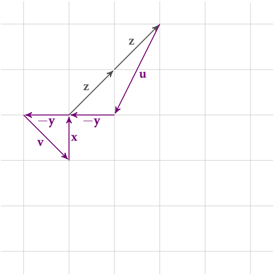


T': in an homogeneous solution e.g., $[a\mathbf{x}, b\mathbf{u}, c\mathbf{y}, d\mathbf{v}]$



- assume $\mathbf{V}' = \mathbf{V}$.
- pick a cycle of G using every transition in T e.g., $\mathbf{x}+\mathbf{z}+\mathbf{u}+\mathbf{y}+\mathbf{v}$
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- as $\mathbf{V} = \mathbf{V}'$, there exists a rational linear combination of cycles of T' with effect Δ e.g., $\Delta = \frac{1}{2}(\mathbf{x}+\mathbf{u}+4\mathbf{y}+\mathbf{v})$
- then $2\Delta = 2(\mathbf{x}+\mathbf{z}+\mathbf{u}+\mathbf{y}+\mathbf{v}) = 2\frac{1}{2}(\mathbf{x}+\mathbf{u}+4\mathbf{y}+\mathbf{v})$
- thus $\mathbf{x}+2\mathbf{z}+\mathbf{u}-2\mathbf{y}+\mathbf{v} = 0$
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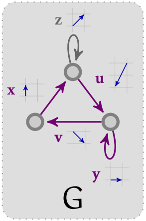
Decreasing Ranks when Unfolding

$T \setminus T'$: not in any homogeneous solution



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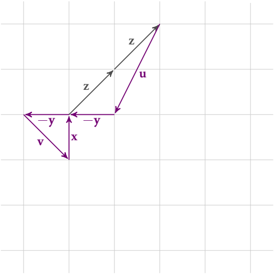

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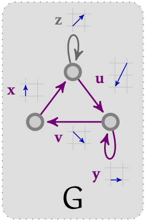

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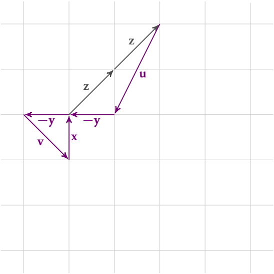

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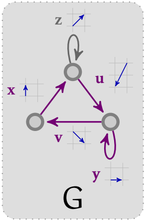

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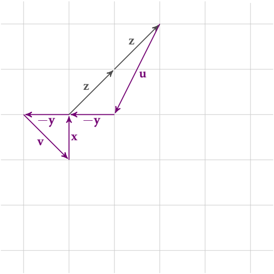

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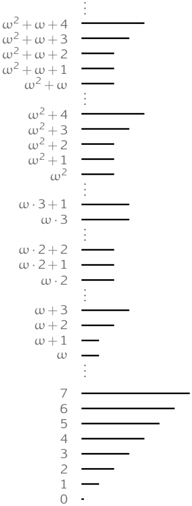
Complexity Upper Bounds

Ordinals



- Cantor normal form for ordinals $\alpha < \varepsilon_0$:

$$\alpha = \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_k} \cdot c_k$$

$$\alpha > \alpha_1 > \cdots > \alpha_k \text{ in CNF}, \qquad 0 < c_1, \ldots, c_k < \omega$$

- norm of ordinals $\alpha < \varepsilon_0$: "maximal constant"

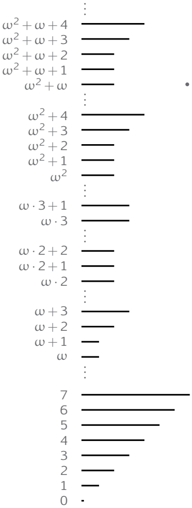
$$N\alpha \stackrel{\text{def}}{=} \max_{1 \leqslant i \leqslant k} (\max(N\alpha_i, c_i))$$

Example

$$N7 = 7 \qquad N(\omega \cdot 3 + 1) = 3$$

$$N(\omega^2 + \omega) = 2 \qquad N(\omega^2 + \omega + 4) = 4$$

Descending Ordinal Sequences

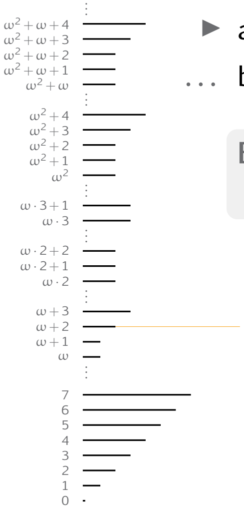


- always finite

… but can be of arbitrary length

Example

Descending Ordinal Sequences



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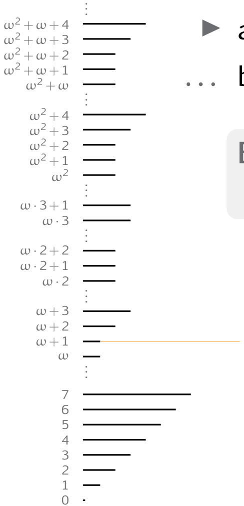
Example

$$\omega + 2$$

Descending Ordinal Sequences



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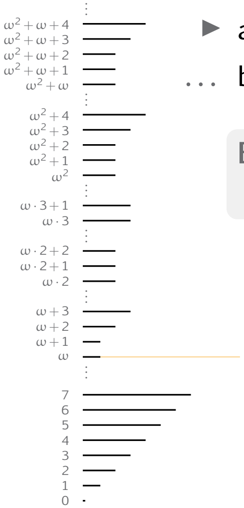
Example

$$\omega + 2 > \omega + 1$$

Descending Ordinal Sequences



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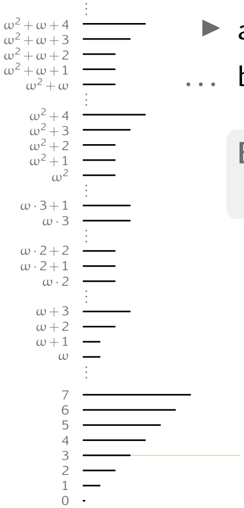
Example

$$\omega+2 > \omega+1 > \omega$$

Descending Ordinal Sequences



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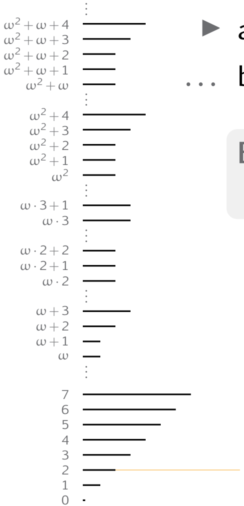
Example

$$\omega+2 > \omega+1 > \omega > 3$$

Descending Ordinal Sequences



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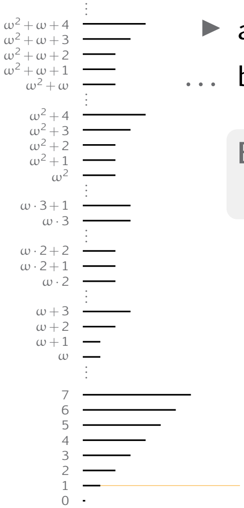
Example

$$\omega+2>\omega+1>\omega>3>2$$

Descending Ordinal Sequences



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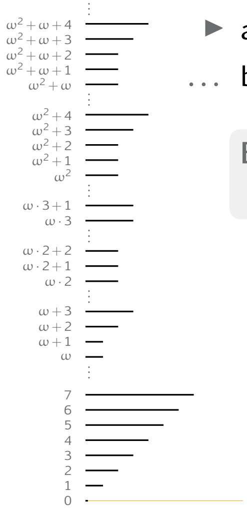
Example

$$\omega+2>\omega+1>\omega>3>2>1$$

Descending Ordinal Sequences



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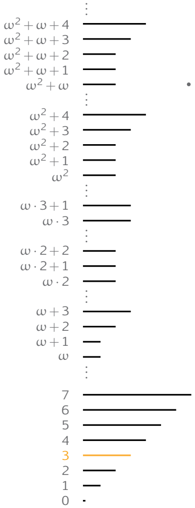

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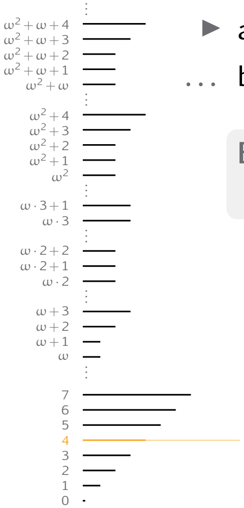

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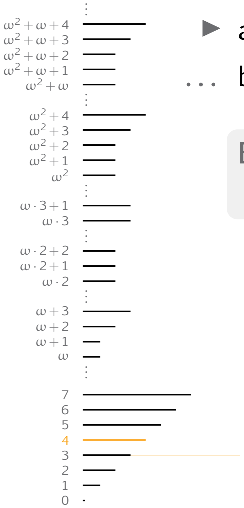
Example

$$\omega + 2 > \omega + 1 > \omega > 4$$

Descending Ordinal Sequences



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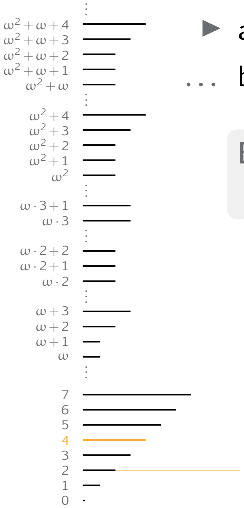
Descending Ordinal Sequences



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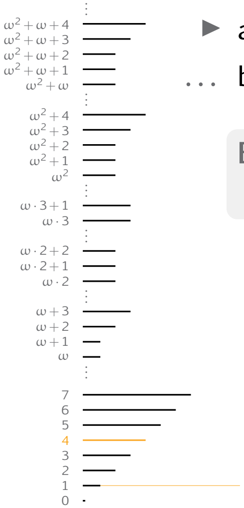

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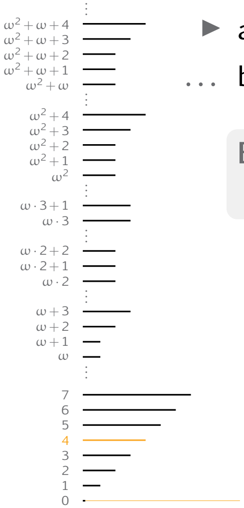

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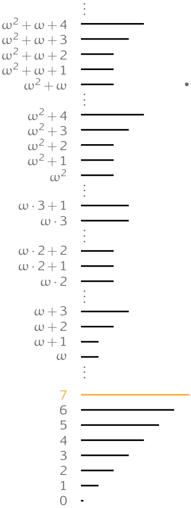

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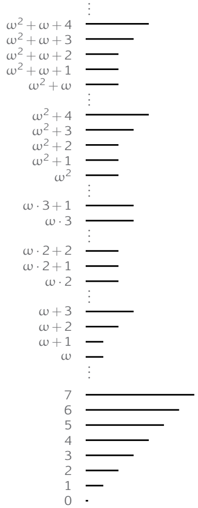
Example

$\omega+2 > \omega+1 > \omega > 7 > 6 > 5 > 4 > 3 > 2 > 1 > 0$

Descending Ordinal Sequences



- $\alpha_0 > \alpha_1 > \dots$ is controlled by $g\colon \mathbb{N} \to \mathbb{N}$ (monotone inflationary) and $n_0 \in \mathbb{N}$ if

$$\forall i\,.\, N\alpha_i \leqslant g^i(n_0)$$

Example $(g(x) = x + 1, n_0 = 2)$

Proposition

Descending sequences of ordinals in $\alpha < \varepsilon_0$ controlled by g and n_0 have a maximal length, noted $L_{g,\alpha}(n_0)$.

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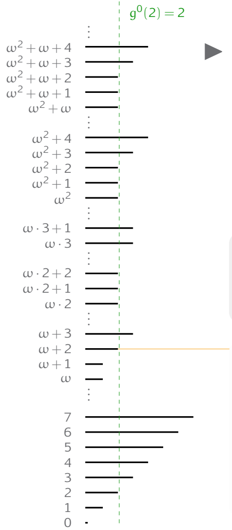

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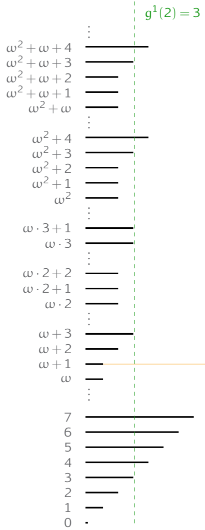

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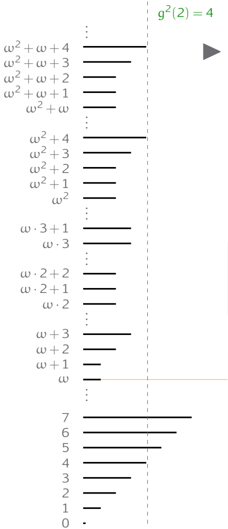

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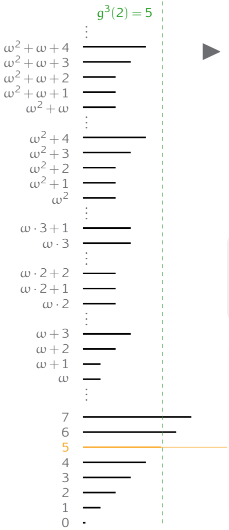

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$$\omega+2 > \omega+1 > \omega > 5$$

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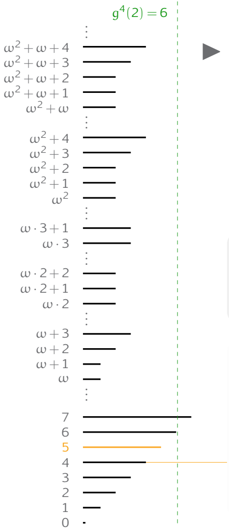

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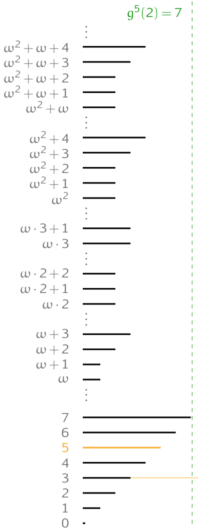

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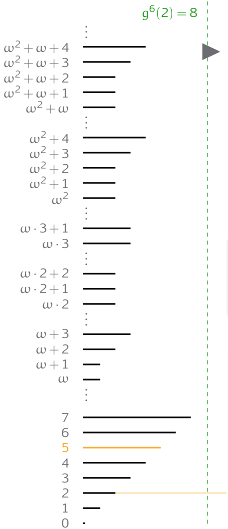

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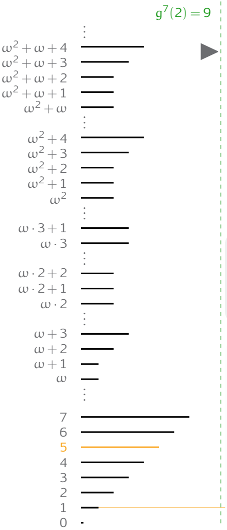

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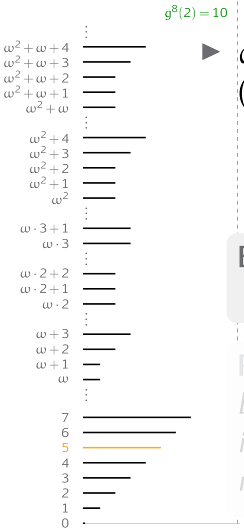

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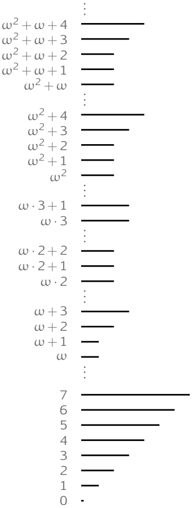

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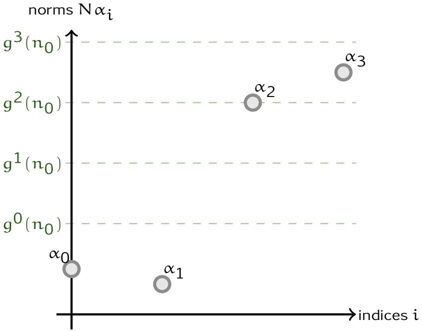
Descent Equation

(g,n_0)-controlled descending sequence $\alpha_0,\alpha_1,\alpha_2,\alpha_3,\ldots$ over an ordinal α:



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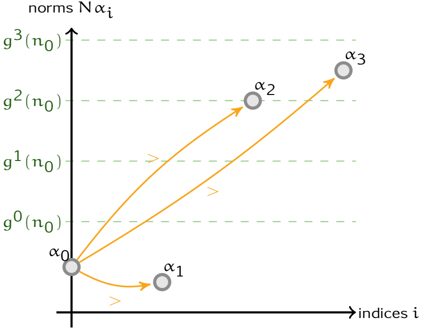

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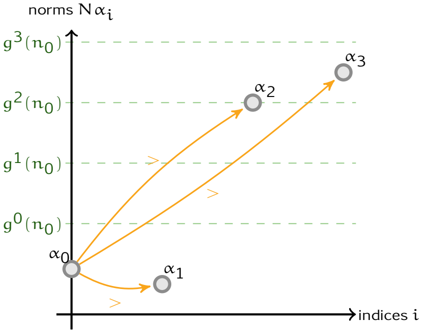

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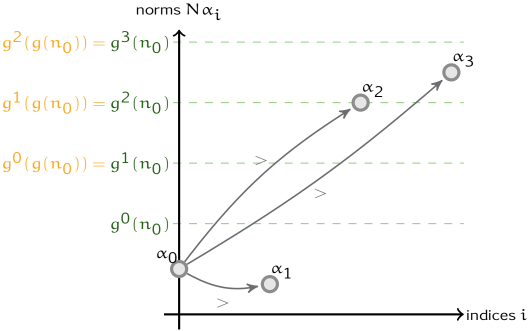

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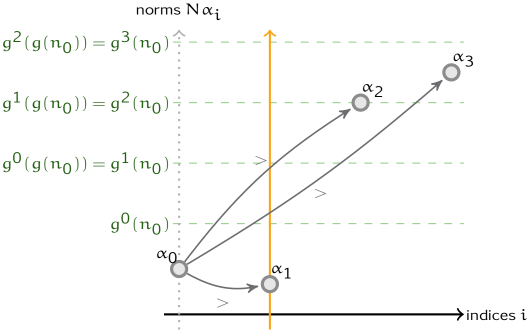

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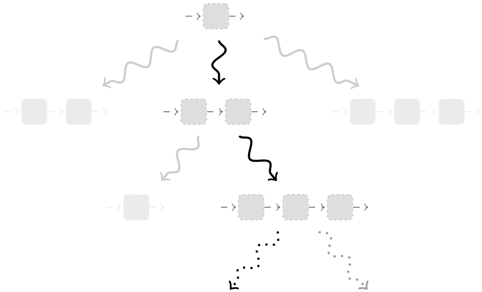
Consequence of (S. '14, '16)

For g elementary, $L_{g,\omega^{d+1}}(n_0) \leqslant F_{d+4}(e(n_0))$ *for some elementary function e.*

The Length of Decomposition Branches

[Leroux & S. '19]



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Corollary

The decomposition tree is of size at most $F_{d+4}(e(n))$ for some elementary function e.

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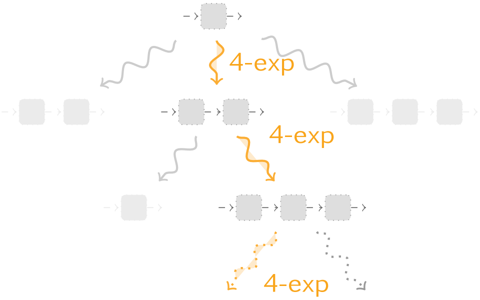

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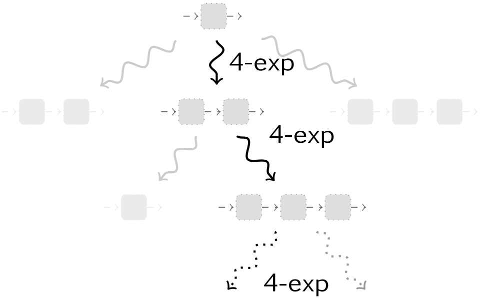

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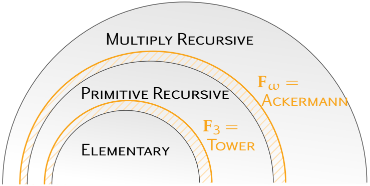

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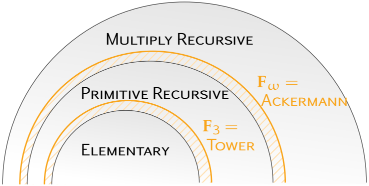

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Theorem ([Leroux '20])

VAS Reachability reduces to bounded VAS Reachability

A Related Problem

labelled VAS transitions carry labels from some alphabet

$L(\mathcal{V}, \textbf{source}, \textbf{target})$ the language of labels in runs from **source** to **target**

$\downarrow L$ the set of scattered subwords of the words in the language L

Example (scattered subword ordering)

$$\mathtt{aba} \leqslant_* \mathtt{baaacabbab}$$

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Downwards Language Inclusion Problem

input: *two labelled VAS* $\mathcal{V}$ *and* $\mathcal{V}'$ *and configurations* $\mathbf{source}, \mathbf{target}, \mathbf{source}', \mathbf{target}'$

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Given a labelled VAS $\mathcal{V}$ and configurations $\mathbf{source}$ and $\mathbf{target}$ and its decomposition, one can construct a finite automaton for $\downarrow L(\mathcal{V}, \mathbf{source}, \mathbf{target})$ in polynomial time.

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The Downwards Language Inclusion is in Ackermann.

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Theorem (Zetzsche'16)

The Downwards Language Inclusion is Ackermann*-hard.*

Perspectives

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   - Tower-hard [Czerwinski et al.'19]
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