Analytic Combinatorics 2/2

Marni Mishna

SFU

Department of Mathematics
Simon Fraser University
Burnaby Canada

EJCIM • "Bordeaux", France

Motivating question #1

In a given regular language over $\{0, 1\}$, what is the proportion of words of length n that have the same number of 0s and 1s? (for n large) What about an alphabet of size k? What about for a context-free language?

Motivating question #2

Which proportion of sequences of n "king" chess moves on $\mathbb{Z}^2$ start and end at the origin, and stay in $\mathbb{N}^2$? (3-D version? arbitrary dimension? other possible moves? regions?)

A Course on Analytic Combinatorics

Objective

Develop tools for analysing the large scale behaviour of combinatorial classes in a **systematic** manner.

Strategy

Examine singularities of multivariable combinatorial generating functions and develop a geometric understanding of its singular structure and then deduce the asymptotic expressions for counting sequences.

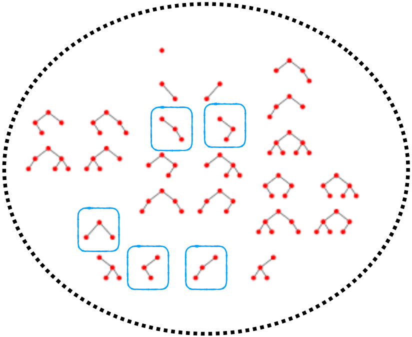
Organization

- I. Combinatorial Framework
  - Combinatorial Calculus
  - Parameters and Extracted Classes
- II. Introductory Singularity Analysis & Asymptotic Expressions
  - Univariate case
  - Multivariate case

Recall the Combinatorial Framework

Combinatorial Classes

A **class** is a set $\mathcal{C}$, and size $|\cdot|$. The number of elements of a given size is finite.



tree $\mapsto z^{\#\text{nodes}}$

$$C(z) = z + 2z^2 + 5z^3 + 14z^4 + 42z^5$$

$$C(z) := \sum_{\gamma \in \mathcal{C}} z^{|\gamma|}$$

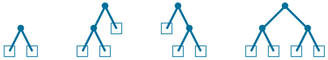
$$= \sum_{n \geq 0} c_n z^n$$

$c_n = \#$ objects of size n

$C(z)$ is the ordinary generating function (OGF) for $\mathcal{C}$

Combinatorial Calculus

	$\mathcal{C}$	Notes	$C(z) = \sum z^{\vert\gamma\vert}$
Epsilon	$\{\epsilon\}$	$\vert\epsilon\vert = 0$	1
Atom	$\{\circ\}$	$\vert\circ\vert = 1$	z
Disjoint Union	$\mathcal{A} + \mathcal{B}$	$\gamma \times \epsilon_{\mathcal{A}}, \gamma \times \epsilon_{\mathcal{B}}$	$A(z) + B(z)$
Cartesian Product	$\mathcal{A} \times \mathcal{B}$	$(\alpha, \beta), \alpha \in \mathcal{A}, \beta \in \mathcal{B}$	$A(z)B(z)$
Power	$\mathcal{A}^k$	$(\alpha_1, \ldots, \alpha_k), \alpha_i \in \mathcal{A}$	$A(z)^k$
Sequence	$\mathrm{Seq}(\mathcal{A}) = \mathcal{A}^*$	$\epsilon + \mathcal{A} + \mathcal{A}^2 + \mathcal{A}^3 + \ldots$	$\frac{1}{1-A(z)}$

Binary Trees $\mathcal{B} := \{\square,$  $, \ldots\}$

$\{\bullet\}$	$\{\square\}$	$\mathcal{B}$	$\equiv$	$\square$	$+$	$\bullet$	$\times$	$\mathcal{B}^2$
$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$		$\downarrow$		$\downarrow$
1	z	$B(z)$	$=$	z	$+$	1	$\cdot$	$B(z)^2$

$$\Longrightarrow \quad B(z) = \frac{1-\sqrt{1-4z}}{2z}$$

Combinatorial Parameters

A **parameter** of a class is a map $\chi : \mathcal{C} \to \mathbb{Z}$
e.g. # 0 in a binary word ; end position (i, j) of a lattice walk

$$C(u, z) := \sum_{\gamma \in \mathcal{C}} u^{\chi(\gamma)} x^{|\gamma|} = \sum_{n \geq 0} \left(\sum_{k \in \mathbb{Z}} c_{k,n} u^k \right) z^n.$$

$c_{k,n}$ = # objects of size n with parameter value k.
$C(u, z) \in \mathbb{N}[u, u^{-1}][[z]]$ Power series with Laurent polynomial coefficients

Example

$\chi(w) = |w|_\circ$ = # ◦s a word in $\{\circ, \bullet\}^*$: $\chi(\circ \bullet \bullet \circ \bullet) = 2$

$$C(u, z) = 1 + (u + 1) z + \left(u^2 + 2u + 1\right) z^2 + \left(u^3 + 3u^2 + 3u + 1\right) z^3 + \ldots.$$

$$C(u, z) = \left(\frac{1}{1 - (z + uz)} \right). \qquad (1)$$

Treat it as a 2-dimensional parameter: $w \mapsto (\chi(w), |w|)$.

Balanced word classes

$$\mathcal{L} = \{\text{binary expansions of } n \mid n \equiv 0 \mod 3.\}$$ Size = length of string

$$\mathcal{L} = \{\epsilon, \overset{0}{0}, 00, 000, \ldots, \overset{3}{11}, 011, 0011, \ldots, \overset{6}{110}, 0110, 00110, \ldots,$$
$$\overset{9}{1001}, 01001, \overset{12}{1100}, 01100, \ldots, \overset{15}{1111}, 01111, \ldots\}$$

S-regular specification: $\mathcal{L} = (0 + (1(01^*0)^*1))^*$

Parameter: $\chi(w) = (|w|_0, |w|_1, |w|) = (\#0s \text{ in } w, \#1s \text{ in } w, |w|)$

Balanced sub-class:

$$\begin{aligned}\mathcal{L}_= &= \{w \in \mathcal{L} \mid \chi(w) = (n, n, 2n), n \geq 0\} \\ &= \{w \in \mathcal{L} \mid \#0s = \#1s\} \\ &= \{1001, 0011, 0110, 1100, 010101, 101010, 11100001, 10011001, \\ &\quad 10000111, 00101101, 01011010, 00111001, 00100111, \ldots\}\end{aligned}$$

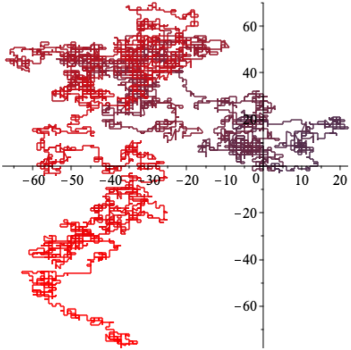
more interesting: $\mathcal{L} \subseteq \{a_1, a_2, \ldots, a_d\}^*$ with $\chi_i(w) = \#$ of a_i in w.

Excursions

$S = \{\uparrow, \downarrow, \leftarrow, \rightarrow\} = ✥$ is a set of steps. Consider walks starting at $(0,0)$ taking steps from S. Unrestricted walks are S-regular:

$$\{\uparrow, \downarrow, \leftarrow, \rightarrow\}^*$$

Define parameter $\chi(w) := $ (endpoint of w, # of steps).



Endpoint is an inherited parameter

$$\sum walk^{✥}_{\mathbb{Z}^2}((0,0) \xrightarrow{n} (k,\ell))\, x^k y^\ell t^n = \frac{1}{1 - t(x + 1/x + y + 1/y)}$$

Excursions are a derived class

$$\mathcal{E} = \{w \in \{\uparrow, \downarrow, \leftarrow, \rightarrow\}^* \mid \chi(w) = (0,0,n)\}$$

Diagonals

The central diagonal maps series expansions to series expansions. e.g.

$$\Delta : K[[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}]][[t]] \to K[[t]].$$

defined as:

$$\Delta F(\mathbf{z}, t) = \Delta \sum_{k \geq 0} \sum_{\mathrm{n} \in \mathbb{Z}^d} f(n_1, n_2, \ldots, n_d, k)\, \mathbf{z}^{\mathrm{n}} t^k := \sum_{n \geq 0} f(n, n, \ldots, n)\, t^n. \tag{2}$$

$$\Delta(z_1^2 z_2 t + 3\mathbf{z}_1 \mathbf{z}_2 \mathbf{t} + 7 z_1 z_2 t^2 + 5\mathbf{z}_1^2 \mathbf{z}_2^2 \mathbf{t}^2) = 3t + 5t^2$$

Example: Multinomials

$$\Delta \frac{1}{1-x-y} = \sum_{n\geq 0} \binom{2n}{n,n} y^n.$$

$$\Delta^{(r,s)} \frac{1}{1-x-y} = \sum_{n\geq 0} \binom{rn+sn}{rn,sn} y^n.$$

$$\Delta^{\mathrm{r}} \frac{1}{1-(z_1+\cdots+z_d)} = \sum_{n\geq 0} \binom{n(r_1+\cdots+r_d)}{nr_1,\ldots,nr_d} z_d^n.$$

Balanced word classes

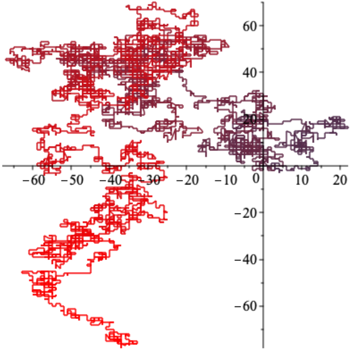
$$\mathcal{L} \subseteq \{0,1\}^2$$

Parameter $\chi(w) = (|w|_0, |w|_1, n) = (\#0s \text{ in } w, \#1s \text{ in } w)$

$$\mathcal{L}_= := \{w \in \mathcal{L} \mid \#0s = \#1s\}$$

$$L_=(y) = \Delta L(x,y)$$

Excursions



Excursions: start and end at $(0,0)$ with steps from $S = ✥$:

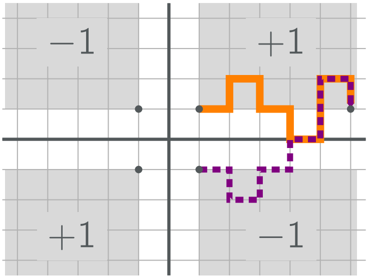
$$\mathcal{E} = \{w \in \{\uparrow, \downarrow, \leftarrow, \rightarrow\}^* \mid \chi(w) = (0,0)\}$$

OGF for excursions:

$$\sum walk_{\mathbb{Z}^2}^{✥}((0,0) \xrightarrow{n} (0,0))\, t^n = [x^0 y^0] \frac{1}{1 - t(x + 1/x + y + 1/y)}$$
$$= \Delta \frac{1}{1 - txy(1/x + x + 1/y + y))}$$

Walks confined to a quadrant - Reflection Principle

$$\sum_{n\geq 0} \text{walk}_{\mathbb{N}^2}^{\leftrightarrow\updownarrow}((0,0) \xrightarrow{n} (0,0))\, t^n$$



$$= [x^1y^1]\frac{xy - x/y + (xy)^{-1} + y/x}{(1 - t(x + 1/x + y + 1/y))}$$

$$= \text{CT}\, \frac{\left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right)}{xy(1 - t(x + 1/x + y + 1/y))}$$

$$= \Delta \frac{xy\left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right)}{1 - txy(x + 1/x + y + 1/y)}$$

$$= \Delta \frac{(x^2 - 1)(y^2 - 1)}{1 - t(x^2y + y + xy^2 + x)}.$$

II. Singularities and Critical Points

Objective

- Systematic methods to determine asymptotic estimates for the number of objects of size n in combinatorial class $\mathcal{C}$.
- Find simple $\Phi(n)$ with $\lim_{n\to\infty} \Phi(n)/c_n = 1$ eg. $\Phi(n) = \gamma k^n n^{-r}$.

Today

We focus on diagonals of multivariable rationals

$$\Delta \frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)} = \Delta \sum_{k\geq 0} \sum_{\mathrm{n}\in\mathbb{Z}^d} f(\mathbf{n}, k)\, \mathbf{z}^{\mathrm{n}} t^k := \sum_{n\geq 0} f(n, n, \ldots, n)\, t^n.$$

Analytic Strategy

- Understand the singularity structure of its OGF $C(t)$
- Relate the singularities of $\frac{G(\mathrm{z},t)}{H(\mathrm{z},t)}$ and $C(t) = \Delta \frac{G(\mathrm{z},t)}{H(\mathrm{z},t)}$.
- Today: Look at the geometry of the variety of points annihilating H.

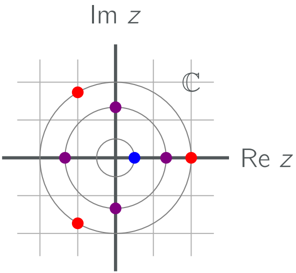
Univariate Singularity

Poles

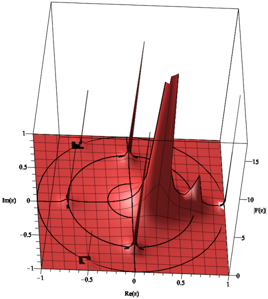
$$F(z) = \frac{1}{(1-z^3)(1-4z)^2(1-5z^4)} = \sum f_n z^n$$

$$= 1 + 8z + 54\,z^2 + 304\,z^3 + 1599\,z^4 + 7928\,z^5 + O\left(z^6\right)$$

$$f_n \sim \mathrm{const}(1/4)^{-n} n^{-1}$$



Poles of $F(z)$

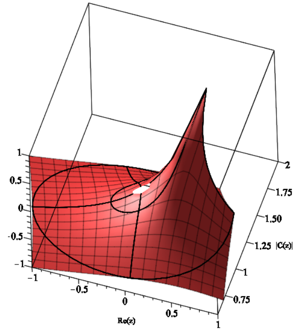


Value of $|F(z)|$

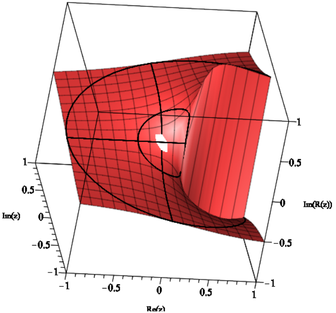
Branch Point Singularity at 1/4

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z} \qquad = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + O(z^7)$$

$$c_n \sim \text{const}(1/4)^{-n} n^{-3/2}$$



$|C(z)|$



$\text{Im}\, C(z)$

Exponential growth and the radius of convergence

$$F(z) = \sum_{n \geq 0} f_n z^n \in \mathbb{R}_{>0}[[z]]$$

$\implies$ there is a positive real valued singular point $\rho \in \mathbb{R}_{>0}$ on the circle of convergence. (Pringshheim)

Exponential growth

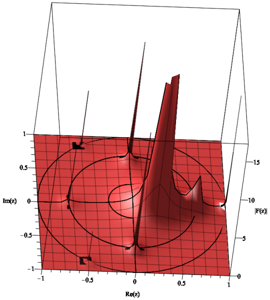
$$\mu := \limsup_{n \to \infty} f_n{}^{1/n}$$

$$\text{“} f_n \sim \kappa \mu^n n^\alpha \text{”}$$

$$ROC(F(z)) = \rho \implies \mu = \rho^{-1}$$

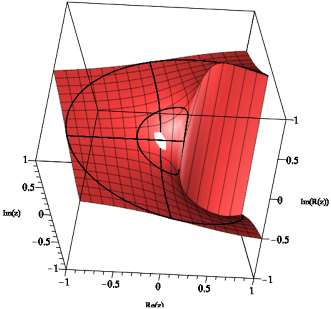
Convergence and the Exponential Growth

The radius of convergence $F(z) = \rho$
$\Longrightarrow$ exponential growth of $f_n = \rho^{-1}$.



$$\frac{1}{(1-z^3)(1-4z)^2(1-5z^4)}$$

$$\lim_{n\to\infty} r_n^{1/n} = 4$$



$$\frac{1-\sqrt{1-4z}}{2z}$$

$$\lim_{n\to\infty} c_n^{1/n} = 4$$

The first principle of coefficient asymptotics

The **location** of singularities of an analytic function determines the **exponential order** of growth of its Taylor coefficients.

We connect the boundary of convergence and exponential growth.

Preview: An analogy

Here is a rough idea of what the multivariable case looks like.

Univariate Rationals

$$F(z) = \frac{G(z)}{H(z)} = \sum f_n z^n$$

$$f_n \sim C\mu^n n^\alpha$$

dominant singularity: $\rho \in \mathbb{C}$ on circle of convergence satisfying $H(\rho) = 0$

$$\mu = |\rho|^{-1}$$

Multivariable Rationals

$$\Delta \frac{G(z_1, \ldots, z_d)}{H(z_1, \ldots, z_d)} = \sum f_{nn\ldots n} z_d^n$$

$$f_{nn\ldots n} \sim C\mu^n n^\alpha$$

minimal critical point: $(\rho_1, \ldots, \rho_d)$ on the boundary of convergence satisfying $H(\rho_1 \ldots, \rho_d) = 0$ + other equations.

$$\mu = |\rho_1 \ldots \rho_d|^{-1}$$

Multivariable Series

Convergence of multivariable series

- View the series as an iterated sum.

$$\sum_{n_d} \left(\dots \left(\sum_{n_1} a(n_1, \dots, n_d) z_1^{n_1} \right) \dots \right) z_d^{n_d}$$

- The domain of convergence, denoted $\mathcal{D} \subseteq \mathbb{C}^d$, is the interior of the set of points where the series converges absolutely.
- The polydisk of a point z is the domain

$$D(\mathbf{z}) = \{\mathbf{z}' \in \mathbb{C}^d : |z_i'| \leq |z_i|, 1 \leq i \leq d\}.$$

- The torus associated to a point is

$$T(\mathbf{z}) = \{\mathbf{z}' \in \mathbb{C}^d : |z_i'| = |z_i|, 1 \leq i \leq d\}.$$

- A domain of convergence is multicircular.

$$\mathbf{z} = (z_1, \dots, z_d) \in \mathcal{D} \implies T(\mathbf{z}) \subseteq \mathcal{D} \implies (\omega_1 z_1, \dots, \omega_d z_d) \in \mathcal{D}, \quad |\omega_k| = 1$$

What is a singularity of G/H?

The set of singularities of $G(z)/H(z)$ is the algebraic variety

$$\mathcal{V} := \{\mathbf{z} : H(\mathbf{z}) = 0\}.$$

minimal points (working definition)

The set of minimal points of a series expansion of F is the set of singular points on the boundary of convergence.

$$\mathcal{V} \cap \partial\mathcal{D}$$

A point $\mathbf{z}$ is **strictly minimal** if $\mathcal{V} \cap D(\mathbf{z}) = \{\mathbf{z}\}$

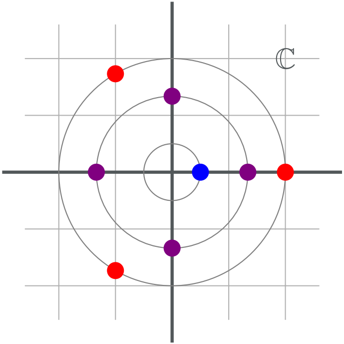
Example: 1D

Definitions

The set of minimal points of a series development of F is the set of singular points on the boundary of convergence.

$$\mathcal{V} \cap \partial\mathcal{D}$$

A point $\mathbf{z}$ is strictly minimal if $\mathcal{V} \cap D(\mathbf{z}) = \{\mathbf{z}\}$



$$H(z) = (1 - z^3)(1 - 4z)^2(1 - 5z^4)$$

$$\partial\mathcal{D} = \{z : |z| = 1/4\},\ \mathcal{V} = \{1/4, 1, w, w^2, (\tfrac{1}{5})^{1/4}\}$$

minimal point: $\mathcal{V} \cap \mathcal{D} = \{1/4\}$, strictly minimal.

Example: $F(x,y) = \frac{1}{1-x-y}$

Taylor expansion: $\sum_{k,\ell} \binom{k+\ell}{k} x^k y^\ell$

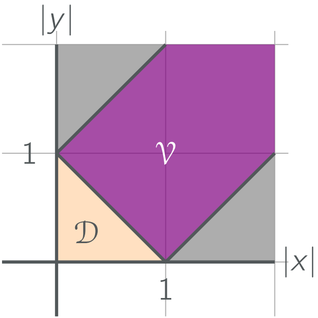
Convergence at (x, y)

$\implies$ convergence at $(|x|, |y|)$

$$F(|x|,|y|) = \frac{1}{1-|x|-|y|} \implies |x|+|y| < 1$$

$$\partial\mathcal{D} = \{(x,y) \in \mathbb{C}^2 \mid |x|+|y| = 1\}$$

$$\mathcal{V} = \{(z, 1-z) \mid z \in \mathbb{C}\}$$



Minimal points $\mathcal{V} \cap \partial\mathcal{D}$

$$\{(z, 1-z) \mid |z| + |1-z| = 1\} = \{(x, 1-x) \mid x \in \mathbb{R}_{>0}\}$$

All strictly minimal.

A first formula for exponential growth for diagonal coefficients

Convergence and exponential growth

Given series $\sum a(\mathbf{n})\,\mathbf{z}^{\mathrm{n}}$ and $\mathbf{z} \in \mathcal{D}$,

$$\sum_{\mathrm{n}\in\mathbb{N}^d} a(\mathbf{n})|z_1|^{n_1}|z_2|^{n_2}\dots|z_d|^{n_d} \text{ is convergent (absolute conv).}$$

$$\Longrightarrow \sum_{n\in\mathbb{N}} a(n,n,\dots,n)\,|z_1|^n|z_2|^n\dots|z_d|^n \text{ is convergent (subseries).}$$

$$= \sum_{n} a(n,n,\dots,n)|z_1 z_2 \dots z_d|^n$$

$\Longrightarrow$ $t = |z_1 z_2 \dots z_d|$ is within the radius of convergence of $\Delta F(\mathbf{z})$.

$$\mu \le \limsup_{n\to\infty} |a(n,n,\dots,n)|^{1/n} \le |z_1 z_2 \dots z_d|^{-1} \quad \text{with } \forall \mathbf{z} \in \overline{\mathcal{D}}$$

$$\le \inf_{(z_1,\dots,z_d)\in\overline{\mathcal{D}}} |z_1 z_2 \dots z_d|^{-1}.$$

Thm: Under conditions of non-triviality, the infimum is reached at a minimal point:

$$\mu = \inf_{\mathrm{z}\in\partial\mathcal{D}\cap\mathcal{V}} |z_1 \dots z_d|^{-1}. \tag{3}$$

Example: Binomials $F(x, y) = (1 - x - y)^{-1}$

Minimal points: $\partial\mathcal{D} \cap \mathcal{V} = \{(x, 1-x) \in \mathbb{R}^2 : 0 < x < 1\}$.

$\mu = \limsup_{n\to\infty} a(n,n)^{1/n} = \inf_{(x,y)\in\partial\mathcal{D}\cap\mathcal{V}} |xy|^{-1} = \inf_{x\in\mathbb{R}:0\le x\le 1} (x(1-x))^{-1} = 4.$

We can consider non-central diagonals.

$$\limsup_{n\to\infty} a_{rn\,sn}^{1/n} = \inf_{(x,y)\in\partial\mathcal{D}} |x^r y^s|^{-1} = \inf_{x\in\mathbb{R}} (x^r(1-x)^s)^{-1}.$$

This is minimized at $x = \frac{r}{r+s}$.
The exponential growth:

$$\mu = \left(\left(\frac{r}{r+s}\right)^r \left(\frac{s}{r+s}\right)^s\right)^{-1}.$$

We got lucky here – we could easily write y in terms of x. What to do in general?

Computing critical points

The height function h

Astuce

We convert the multiplicative minimization to a linear minimization using logarithms.

To minimize $|z_1 \dots z_d|^{-1}$, minimize:

$$-\log|z_1 \dots z_d| = \underbrace{-\log|z_1| - \cdots - \log|z_d|}_{\text{linear in } \log|z_i|}$$

Define a function $h : \mathcal{V}^* \to \mathbb{R}$:

$\mathcal{V}^* = \mathcal{V} \setminus \{z : z_1 \dots z_d \neq 0\}$

$$(z_1, \dots, z_d) \mapsto -\log|z_1| - \cdots - \log|z_d|.$$

The map h is smooth $\implies$ minimized at its critical points. When $r = (1, \dots, 1)$, the critical points are solutions to the critical point equations:

$$H(\mathbf{z}) = 0, \quad z_1 \frac{\partial H(\mathbf{z})}{\partial z_1} = z_j \frac{\partial H(\mathbf{z})}{\partial z_j}, 2 \leq j \leq d.$$

Critical points

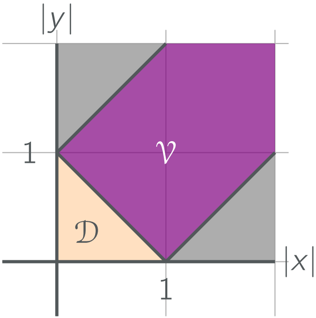
Critical points are potential locations of minimizers of $|z_1 \dots z_d|^{-1}$.
In the most straightforward cases it suffices to compare the values of this product and select the critical point that is the global minimizer.

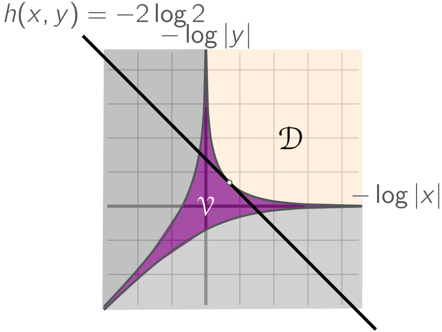
- A critical point is strictly minimal if it is on the boundary of convergence of the series.
- In these generating functions the asymptotics is driven by a finite number of isolated minimal points. Simplest case.

Visualize Critical Points

Critical points of $(1-x-y)^{-1}$ for $r=(1,1)$

1. $\rho \in \mathcal{V} \cap \partial\mathcal{D}$ — $\rho = (1/2, 1/2)$
2. minimize $h(x,y) = -\log|x| - \log|y|$ — $h(x,y) = -2\log 2$
3. $\mu = |\rho_1\rho_2|^{-1}$ — $\lim_{n\to\infty} \binom{2n}{n}^{1/n} = 4$



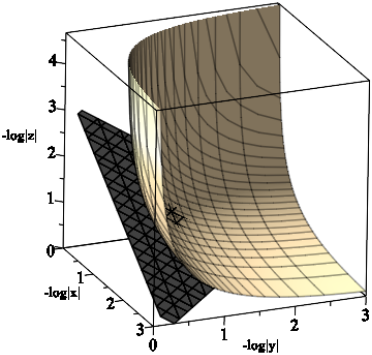


$$(x,y) \mapsto (-\log|x|, -\log|y|)$$

Trinomial $(1 - x - y - z)^{-1}$

Critical points

1. $\rho \in \mathcal{V} \cap \partial\mathcal{D}$ $\qquad \rho = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
2. minimize $h(x, y, z) = -\log|x| - \log|y| - \log|z|$ $\qquad h(x, y, z) = 3\log 3$
3. $\mu = |\rho_1 \rho_2 \rho_3|^{-1}$ $\qquad \lim_{n\to\infty} \binom{3n}{n,n,n}^{1/n} = 27$



Non-central diagonals

If we want a non-central diagonal, we want to minimize

$$|z_1^{r_1} \dots z_d^{r_d}|^{-1} \quad in \quad \partial\mathcal{D} \cap \mathcal{V}.$$

Instead take height function here is

$$(z_1, \dots, z_d) \mapsto -r_1 \log|z_1| - \dots - r_d \log|z_d|.$$

The equations change. For example, in 2D, diagonal (r, s), solve the equations:

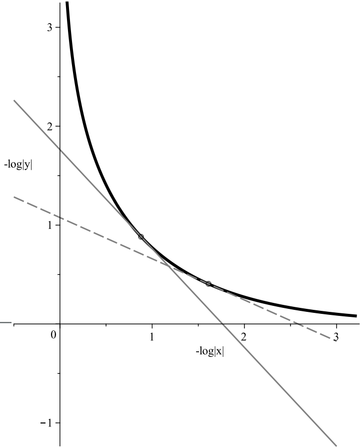
$$H(x, y) = 0, \quad s\,x \frac{\partial H(x, y)}{\partial x} = r\,y \frac{\partial H(x, y)}{\partial y}.$$

Critical point depends on the diagonal ray

Delannoy Numbers

$d(rn, sn) := [x^{rn}y^{sn}](x + y + xy)^n$

(r, s)	$-r \log\lvert x\rvert - s \log\lvert y\rvert$	ρ
$(1, 1)$	———	$(\frac{1}{\sqrt{2}-1}, \frac{1}{\sqrt{2}-1})$
$(5, 2)$	— — —	$(\frac{1}{5}, \frac{2}{3})$



Summary: To Find Critical Points

Given: $G(x,y)/H(x,y) = \sum f_{k,\ell}\, x^j y^k$, irreducible H
$(r,s) \in \mathbb{R}^2_{>0}$
Determine: $\mu = \limsup_{n\to\infty} f_{rn,sn}^{1/n}$, critical points ρ

- Find solutions $\{\rho\}$ to the (r,s)-critical point equations.
  Hint: Find Gröbner basis of
  ```
  [H, s*x*diff(H, x)-r*y*diff(H,y)]
  ```
- Ensure $T(\rho) \subset \partial\mathcal{D}$
- Set $\mu = \min|\rho_1 \dots \rho_d|^{-1}$ among those solutions with no 0 coordinate.
- We use the set of such ρ to find the sub-exponential growth (tomorrow)
- Nontriviality requirement: ρ to be smooth as *a function of* (r,s) near where you want it.

Balanced Binary Words

Let $\mathcal{L}$ =Binary words over $\{0,1\}$ with no run of 1s of length 3.

$$\mathcal{L} = (\epsilon + 1 + 11) \cdot (0 \cdot (\epsilon + 1 + 11))^*$$

Parameter: $\chi(w) = (|w|_0, |w|_1)$

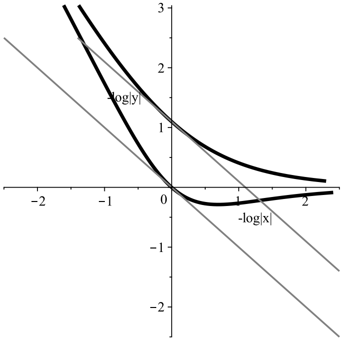
$$\mathcal{L}_= = \{w \in \mathcal{L} \mid \chi(w) = (n,n)\}$$

$$L_=(y) = \Delta \frac{1 + x + x^2}{1 - y(1 + x + x^2)}$$

- GB of Critical point equations: $[x^2 - 1, x + 3y - 2]$
- two solutions: $(1, 1/3)$ $(-1, 1)$
- $\mu = \min |\rho_1 \ldots \rho_d|^{-1}$ $(1, 1/3) \mapsto 3$ $(-1, 1) \mapsto 1$
- BUT $(1,1) \in T(-1,1) \implies (-1,1)$ is not a minimal point because $(1,1)$ outside of domain of convergence.

$$[y^n]L_=(y) \to \kappa 3^n n^\alpha$$

Visualize the boundary



Simple Excursions

Let $\mathcal{E}$ be the set of simple excursions in the entire plane, that is walks that start and end at the origin, taking unit steps $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$

$$e(n) = [x^0 y^0] \left(x + 1/x + y + 1/y\right)^n$$

We can deduce:

$$E(z) = \Delta \frac{1}{1 - zxy\left(x + \frac{1}{x} + y + \frac{1}{x}\right)}$$

Any critical point $\rho = (x, y, z)$ will have $z = \frac{1}{xy(x+1/x+y+1/y)}$ from $H = 0$.
Critical points: $(1, 1, 1/4), (-1, -1, -1/4)$

$$e(2n)^{1/2n} = \inf_{\rho \in \partial\mathcal{D} \cap \mathcal{V}} |xyz|^{-1} = \inf_{0 \leq x, y \leq 1} |x + 1/x + y + 1/y| = 4^2$$

Excursions for any finite step set

This phenomena is general. Let $\mathcal{S}$ be any weighted finite 2D step set

$$S(x, y) = \sum_{(j,k)\in\mathcal{S}} w(j, k)\, x^j y^k$$

$$e(n) = [x^0 y^0] S(x, y)^n$$

We can deduce:

$$E(z) = \Delta \frac{1}{1 - zxyS(1/x, 1/y)}$$

Any critical point $\rho = (x, y, z)$ will have $z = \frac{1}{xyS(1/x,1/y)}$ from $H = 0$.

$$\limsup_{n\to\infty} e(n)^{\frac{1}{n}} = \inf_{\rho\in\partial\mathcal{D}\cap\mathcal{V}} |xyz|^{-1} = \inf_{\rho\in\partial\mathcal{D}} |S(1/x, 1/y)|$$

The minimum is found using the critical point eqn.

What if H factors?

Suppose H factors nontrivially into squarefree factors:

$$H = H_1 \dots H_k$$

- CASE A: $H_j(\rho) = 0 \implies H_k(\rho) \neq 0$ for $j \neq k$: OK.
- CASE B: Must decompose $\mathcal{V}$ into strata, and find critical points for each stratum independently. **Important to keep track of the co-dimension of the stratum for later.**

Walks in the quarter plane that end anywhere

Let $\mathcal{S} = \{\nwarrow, \rightarrow, \downarrow\}$. $\mathcal{T}$= walks start at $(0,0)$ end anywhere. Using a reflection principle argument:

$$T(z) = \Delta \frac{\left(1 - y^2/x + y^3 - x^2y^2 + x^3 - x^2/y\right)}{(1 - zxy(1/x + x/y + y))\,(1-x)(1-y)} \tag{4}$$

Critical points

We divide $\mathcal{V}_H$ into strata and we determine critical points from each of them.

Image of $\mathcal{V}$ under $(x, y, z) \mapsto (|x|, |y|, |z|)$

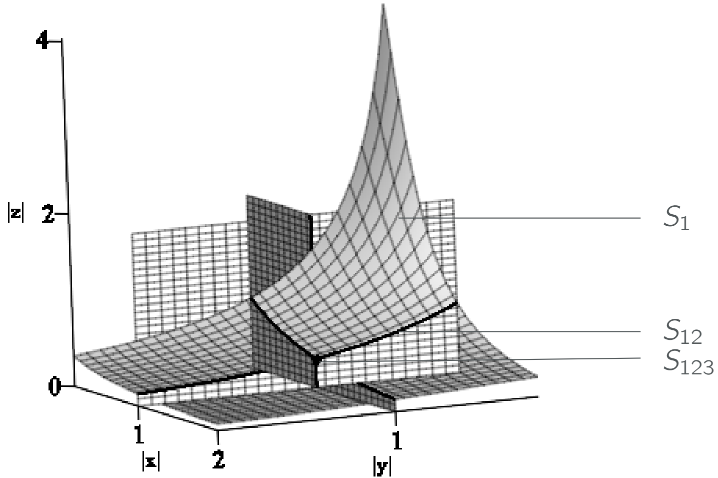
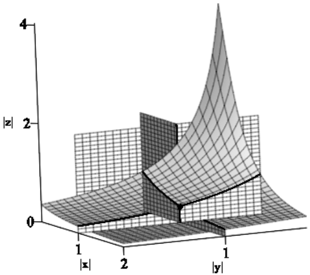
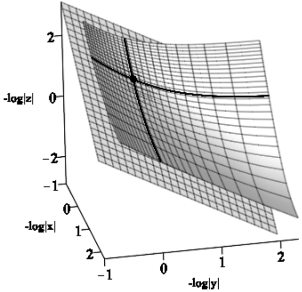


Image of $\mathcal{V}$ under $(x, y, z) \mapsto (-\log|x|, -\log|y|, -\log|z|)$





Stratum	Critical points	value of $\lvert xyz\rvert^{-1}$
S_1	$(w^2, w, w/3)$, $(w, w^2, w^2/3)$	$1/3$
S_{12}		
S_{23}		
S_{123}	$(1, 1, 1/3)$	$1/3$

A lattice path enumeration problem

Let $\mathcal{S} = \{\nwarrow, \rightarrow, \downarrow\}$. $\mathcal{T}$= walks start at $(0,0)$ end anywhere. Using a reflection principle argument:

$$T(z) = \Delta \frac{\left(1 - y^2/x + y^3 - x^2y^2 + x^3 - x^2/y\right)}{\left(1 - zxy(1/x + x/y + y)\right)(1-x)(1-y)}$$

We conclude: Three critical points:

$$(w, w^2, w^2/3), (w^2, w, w/3), (1, 1, 1/3)$$

(Potential for periodicity..)

Exponential growth: $t_n \sim C3^n n^\alpha$.
Next challenge: find C, α.

Next Steps..

Determine **how** each contributing critical point modulates the dominant exponential term by a subexponential factor.

Summary

Diagonal Asymptotics

Given:

$$F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) = \sum f(\mathbf{n})\mathbf{z}^{\mathbf{n}}$$

Determine the asymptotics of $f(n, n, \ldots, n)$ as $n \to \infty$

- Singular Variety $\mathcal{V} = \{\mathbf{z} \mid H(\mathbf{z}) = 0\}$
- Minimal Points: $\partial\mathcal{D} \cap \mathcal{V}$
- Critical points minimize: $|\rho_1 \ldots \rho_d|^{-1}$ (with value μ, say)
- Minimal critical point ρ contained in both

$$\underbrace{-\log|z_1| - \cdots - \log|z_d| = \log\mu}_{\text{a hyperplane}} \qquad \rho \in \partial\mathcal{D} \cap \mathcal{V}$$

Balanced Binary Words

Let $\mathcal{L}$ =Binary words over $\{0,1\}$ with no run of 1s of length 3.

$$\mathcal{L} = (\epsilon + 1 + 11) \cdot (0 \cdot (\epsilon + 1 + 11))^*$$

Parameter: $\chi(w) = (|w|_0, |w|_1)$

$$\mathcal{L}_= = \{w \in \mathcal{L} \mid \chi(w) = (n,n)\}$$

$$L_=(y) = \Delta \frac{1+x+x^2}{1-y(1+x+x^2)}$$

- GB of Critical point equations: $[x^2-1, x+3y-2]$
- two solutions: $(1,1/3)$ $(-1,1)$
- $\mu = \min |\rho_1 \ldots \rho_d|^{-1}$ $(1,1/3) \mapsto 3$ $\qquad (-1,1) \mapsto 1$
- BUT $(1,1) \in T(-1,1) \implies (-1,1)$ is not a minimal point because $(1,1)$ outside of domain of convergence.

$$[y^n]L_=(y) \to \kappa 3^n n^\alpha$$

First Principle of Coefficient Asymptotics

The **location** of singularities of an analytic function determines the **exponential order** of growth of its Taylor coefficients.

We connect the boundary of convergence and exponential growth.

Second Principle of Coefficient Asymptotics

The **nature** of the singularities determines the way the dominant exponential term in coefficients is modulated by a subexponential factor.

Nature = geometry of the singular variety at the critical point.

Bibliography

1. *Analytic Combinatorics in Several Variables.* Pemantle and Wilson Cambridge University Press
2. *Analytic Combinatorics in Several Variables: Effective Asymptotics and Lattice Path Enumeration.*
3. *Analytic Combinatorics: A Multidimensional Approach.* Marni Mishna, CRC Press

MERCI BEAUCOUP