Reachability in VAS is not Elementary

Jérôme Leroux (LaBRI : CNRS & Univ. Bordeaux & Bordeaux-INP)
Joint work with W. Czerwiński, S. Lasota, R. Lazić and F. Mazowiecki.
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Vector addition systems with states (VASS)

(0, 1, -1) → (0, 0, 0) → (0, -1, 2)

States p, q

Actions (0, 1, -1), (0, 0, 0), (1, 0, 0), (0, -1, 2)

Dimension 3

Configurations p(x, y, z), q(x, y, z) with x, y, z ∈ N

Runs p(0, 0, 1) → p(0, 1, 0) → q(0, 1, 0)

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Vector addition systems with states (VASS)

States

$p, q$

Actions

$(0, 0, 0), (0, 1, -1), (0, 0, 0), (0, -1, 2)$

Dimension 3

Configurations $p(x, y, z), q(x, y, z)$ with $x, y, z \in \mathbb{N}$

Runs $p(0, 0, 1) \rightarrow p(0, 1, 0) \rightarrow q(0, 1, 0)$ also written $p(0, 0, 1) \ast - \rightarrow q(0, 1, 0)$
States
$p, q$

Actions
$(0, 1, -1), (0, 0, 0), (1, 0, 0), (0, -1, 2)$
Vector addition systems with states (VASS)

States

$p, q$

Actions

$(0, 1, -1), (0, 0, 0), (1, 0, 0), (0, -1, 2)$

Dimension

3
Vector addition systems with states (VASS)

States
$p$, $q$

Actions
$(0, 1, -1), (0, 0, 0), (1, 0, 0), (0, -1, 2)$

Dimension
3

Configurations
$p(x, y, z), q(x, y, z)$ with $x, y, z \in \mathbb{N}$
Vector addition systems with states (VASS)

States
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Dimension
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Configurations
$p(x, y, z), q(x, y, z)$ with $x, y, z \in \mathbb{N}$

Runs
$p(0, 0, 1) \rightarrow p(0, 1, 0) \rightarrow q(0, 1, 0)$ also written $p(0, 0, 1)^* q(0, 1, 0)$
Petri nets

Model of concurrency with extensive applications in modelling and analysis of
• hardware and software,
• database systems,
• chemical, biological and business processes.

Petri nets $\leftrightarrow$ VASS

Reachability in VAS is not Elementary
Petri nets

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Petri nets

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Petri nets ←→ VASS

Jérôme Leroux (LaBRI : CNRS & Univ. Bordeaux & Bordeaux-INP)

Reachability in VAS is not Elementary
Decision problems

Reachability problem:

**Given:** a VASS \( V \)

**Decide:** whether \( p(0, \ldots, 0) \xrightarrow{*} q(0, \ldots, 0) \) with \( p \) initial and \( q \) final?
Decision problems

Reachability problem:

\textbf{Given:} a VASS $V$

\textbf{Decide:} whether $p(0, \ldots, 0) \rightarrow^* q(0, \ldots, 0)$ with $p$ initial and $q$ final?

Coverability problem:

\textbf{Given:} a VASS $V$

\textbf{Decide:} whether exists $v$ s.t. $p(0, \ldots, 0) \rightarrow^* q(v)$ with $p$ initial and $q$ final?
Decision problems

Reachability problem:

**Given:** a VASS $V$

**Decide:** whether $p(0, \ldots, 0) \xrightarrow{*} q(0, \ldots, 0)$ with $p$ initial and $q$ final?

Coverability problem:

**Given:** a VASS $V$

**Decide:** whether exists $v$ s.t. $p(0, \ldots, 0) \xrightarrow{*} q(v)$ with $p$ initial and $q$ final?

Many problems reduce to reachability or coverability

- Formal languages
- Logic
- Concurrent systems
- Process calculi,...
Reachability state of the art

1976
Expspace lower bound (Lipton)

1981
Decidable (Mayr)

1982
Decidable (Kosaraju)

1992
Decidable (Lambert)

2009-2012
Decidable with Presburger inductive invariants (Leroux)

2015-2019
$F_\omega$ upper bound (Leroux and Schmitz)

2019
$F_3$ lower bound

(Czerwiński, Lasota, Lazić, Leroux and Mazowiecki)

Coverability
Rackoff 1978
Exspace-complete
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Fourier-Stieltjes convolution
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Coverability
Rackoff 1978
Expspace-complete
Outline

1. Large Configurations
2. Witness of Unreachability
3. Counter Programs
4. Lower Bounds
Large Configurations
The Hopcroft-Pansiot 1979 Example

Exponential behavior
The Hopcroft-Pansiot 1979 Example

Exponential behavior

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The Hopcroft-Pansiot 1979 Example

Exponential behavior
Fast Growing Functions

\[ F_0(n) = n + 1 \]
\[ F_{d+1}(n) = F_d^{n+1}(n) \]
Fast Growing Functions

\[ F_0(n) = n + 1 \]
\[ F_{d+1}(n) = F_{d+1}^{n+1}(n) \]

\[ F_1(n) = 2n + 1 \]
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\[ F_2(n) = 2^{n+1}(n + 1) - 1 \]
Fast Growing Functions

\[ F_0(n) = n + 1 \]
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\[ F_3(n) \sim Tower(n) \]

...
Fast Growing Functions

\[ F_0(n) = n + 1 \]
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\[ F_1(n) = 2n + 1 \]
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\[ F_3(n) \sim \text{Tower}(n) \]
...

\[ F_d \] weakly computable with VASSes in \( \text{dim } d + 1 \) (Mayr and Meyer 1981).
Outline

1. Large Configurations
2. Witness of Unreachability
3. Counter Programs
4. Lower Bounds
Witness of Unreachability
Presburger Arithmetic

$\text{FO}(\mathbb{N}, +)$

\[ \phi(x) := \exists k \ x = k + k \]
\[ \phi(x) := x = x + x \]
\[ \phi(x, y) := \exists k \ y = x + k \]
\[ \phi(x) := \forall k \ k \leq x \land k \neq x \implies k = 0 \]

Theorem (Ginsburg and Spanier 1966)

A subset of $\mathbb{N}^d$ is Presburger (definable) if, and only if, it is semilinear.

I.e. a finite union of sets of the form $\vec{b} + \mathbb{N}\vec{p}_1 + \cdots + \mathbb{N}\vec{p}_k$
**Presburger Arithmetic**

\[ \text{FO}(\mathbb{N}, +) \]

\[ 2\mathbb{N} \quad \phi(x) := \exists k \ x = k + k \]
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Presburger Arithmetic

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\[
\begin{align*}
2\mathbb{N} & \quad \phi(x) := \exists k \ x = k + k \\
\{0\} & \quad \phi(x) := x = x + x \\
\{(x, y) & \in \mathbb{N} \times \mathbb{N} \mid x \leq y\} & \quad \phi(x, y) := \exists k \ y = x + k
\end{align*}
\]
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Presburger Arithmetic

$\text{FO}(\mathbb{N}, +)$

$2\mathbb{N}$ \hspace{1cm} $\phi(x) := \exists k \ x = k + k$

$\{0\}$ \hspace{1cm} $\phi(x) := x = x + x$

$\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\}$ \hspace{1cm} $\phi(x, y) := \exists k \ y = x + k$

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$2\mathbb{N}$ \hspace{1cm} $0 + \mathbb{N}2$
Presburger Arithmetic

**FO(ℕ, +)**

\[
2\mathbb{N} \quad \phi(x) := \exists k \ x = k + k
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\[
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A subset of \(\mathbb{N}^d\) is Presburger (definable) if, and only if, it is *semilinear*.

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\[
2\mathbb{N} \quad 0 + \mathbb{N}2
\]

\[
\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\} \quad (0, 0) + \mathbb{N}(0, 1)
\]
Presburger Witness of Unreachability

A witness of unreachability $C$ for a VASS is a set of configurations such that:

- $q(0,\ldots,0) \in C$ for every initial state $q$.
- $c \rightarrow c' \land c \in C \Rightarrow c' \in C$.
- $q(0,\ldots,0) \not\in C$ for every final state $q$.

Theorem (Leroux 2009-2012)

For every VASS:

- Either $p(0,\ldots,0) \not\rightarrow q(0,\ldots,0)$ with $p$ initial and $q$ final.
- Or there exists a Presburger witness of unreachability.
A witness of unreachability $C$ for a VASS is a set of configurations such that:

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**Theorem (Leroux 2009-2012)**

For every VASS:

- Either $p(0, \ldots, 0) \rightarrow^* q(0, \ldots, 0)$ with $p$ initial and $q$ final
- Or there exists a Presburger witness of unreachability
Outline

1. Large Configurations
2. Witness of Unreachability
3. Counter Programs
4. Lower Bounds
Counter Programs
Counter Programs

- Operations over bounded counters $\bar{x} \in \{0, \ldots, B\}$:
  - $\bar{x} += 1$
  - $\bar{x} - = 1$
  - $\bar{x}$ zero?
  - $\bar{x}$ max?

- Operations over unbounded counters $x \in \mathbb{N}$:
  - $x += 1$
  - $x - = 1$

- Non deterministic loop
Counter Programs

• Operations over bounded counters $\bar{x} \in \{0, \ldots, B\}$:
  $\bar{x} += 1$
  $\bar{x} -= 1$
  zero? $\bar{x}$
  max? $\bar{x}$

• Operations over unbounded counters $x \in \mathbb{N}$:
  $x += 1$
  $x -= 1$

• Non deterministic loop

Jérôme Leroux (LaBRI : CNRS & Univ. Bordeaux & Bordeaux-INP)

Reachability in VAS is not Elementary
Examples

\[ \overline{a} \mathrel{+}= 2 \quad \rightarrow \quad \overline{a} \mathrel{+}= 1 \]
\[ \overline{a} \mathrel{+}= 1 \]

\[ x \mathrel{+}= 2 \quad \rightarrow \quad x \mathrel{+}= 1 \]
\[ x \mathrel{+}= 1 \]
Examples

\[ \bar{a} \mathrel{+}= 2 \quad \rightarrow \quad \bar{a} \mathrel{+}= 1 \]
\[ \bar{a} \mathrel{+}= 1 \]

\[ x \mathrel{+}= 2 \quad \rightarrow \quad x \mathrel{+}= 1 \]

assert \( \bar{a} = 2 \quad \rightarrow \quad \text{zero? } \bar{a} \]
\[ \bar{a} \mathrel{+}= 2 \]
Examples

\[ \bar{a} += 2 \rightarrow \bar{a} += 1 \quad \bar{a} += 1 \quad x += 2 \rightarrow x += 1 \]

assert \( \bar{a} = 2 \rightarrow \bar{a} -= 2 \quad \text{zero? } \bar{a} \quad \bar{a} += 2 \]

loop

\[ \bar{a} := 2 \rightarrow \bar{a} -= 1 \quad \text{zero? } \bar{a} \quad \bar{a} += 2 \]
Reachability Problems

A run is **complete** if it starts and ends with zero in every counter.
Reachability Problems

A run is **complete** if it starts and ends with zero in every counter.

Reachability problem (for counter programs):

**GIVEN:** A counter program and a bound \( B \).

**DECIDE:** Does it have a complete run?
Reachability Problems

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Reachability problem (for VASS):

**GIVEN:** a VASS $V$.

**DECIDE:** whether $p(0, \ldots, 0) \xrightarrow{\ast} q(0, \ldots, 0)$ with $p$ initial and $q$ final?
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- **DECIDE**: whether $p(0, \ldots, 0) \rightarrow^* q(0, \ldots, 0)$ with $p$ initial and $q$ final?
Reachability in VAS is not Elementary
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Counter Programs $\rightarrow$ VASS

1. $\overline{i} += 1$
2. loop
3. $x += 1$
4. $\overline{i} += 1$
5. $\overline{i} -= 1$
6.

with $B = 2$. 
counter Programs $\rightarrow$ VASS

1: $\overline{i} += 1$
2: loop
3: $x += 1$
4: $\overline{i} += 1$
5: $\overline{i} -= 1$
6: with $B = 2$.
Outline

1. Large Configurations
2. Witness of Unreachability
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Lower Bounds
Implement with a counter program:

\[ n \cdot \prod_{1 \leq i < B} \frac{i + 1}{i} = n \cdot B \]
A weak multiplier by $\frac{3}{2}$

**loop**

\[
x -\!\!=\!\!= 2 \quad x' +\!\!=\!\!= 3
\]

**loop**

\[
x' -\!\!=\!\!= 1 \quad x +\!\!=\!\!= 1
\]
A weak multiplier by $\frac{3}{2}$

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<td>$x$</td>
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<td>(\frac{3}{2})14 = 21</td>
<td>0</td>
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</table>
A weak multiplier by $\frac{3}{2}$

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\[ \frac{3}{2} \times 14 = 21 \]

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<tr>
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\[ \frac{3}{2} \times 15 > 22 \]

loop
\[ x \leftarrow 2 \quad x' \rightarrow 3 \]

loop
\[ x' \leftarrow 1 \quad x \rightarrow 1 \]
A weak multiplier by $\frac{i+1}{i}$

\[
\begin{align*}
\text{loop} & \quad x \leftarrow \bar{i} & x' \leftarrow \bar{i} + 1 \\
\text{loop} & \quad x' \leftarrow 1 & x \leftarrow 1
\end{align*}
\]
A weak multiplier by $\frac{i+1}{i}$

\[
\text{loop} \\
x \leftarrow 1 \\i \leftarrow 1 \\a \leftarrow 1 \\
\text{zero? } \i \\
\text{loop} \\
\i \leftarrow 1 \\
\a \leftarrow 1 \\
\text{zero? } \a
\]

\[
\text{loop} \\
x \leftarrow i \\
x' \leftarrow i + 1 \\
\text{loop} \\
x' \leftarrow 1 \\
x \leftarrow 1
\]
A weak multiplier by $\frac{i+1}{i}$

\[
\begin{align*}
\text{loop} & \\
& x \leftarrow i \quad x' \leftarrow i + 1
\end{align*}
\]

\[
\begin{align*}
\text{loop} & \\
& x' \leftarrow 1 \quad x \leftarrow 1
\end{align*}
\]
Implementing $n \cdot \prod_{1 \leq i < B} \frac{i+1}{i} = n \cdot B$
Implementing $n \cdot \prod_{1 \leq i < B} \frac{i+1}{i} = n \cdot B$

\[
\begin{align*}
x &= x + 1 \quad y &= y + 1 \\
\text{loop} & \quad \begin{cases}
x &= x + 1 \quad y &= y + 1 \\
\bar{i} &= \bar{i} + 1
\end{cases} \quad \text{init } x \text{ and } y \text{ to some } n \geq 1 \\
\text{loop} & \quad \begin{cases}
x &= x - \bar{i} \quad x' &= x' + \bar{i} + 1 \\
\text{loop} & \quad \begin{cases}
x' &= x' - 1 \quad x &= x + 1 \\
\bar{i} &= \bar{i} + 1
\end{cases}
\end{cases} \quad \text{weak multiplier by } \frac{\bar{i}+1}{\bar{i}} \\
\text{max? } \bar{i} & \quad \begin{cases}
x &= x - \bar{i} \quad y &= y - 1 \quad \text{multiplication checker}
\end{cases}
\text{loop} & \quad \begin{cases}
\bar{i} &= \bar{i} - 1 \quad \text{reset } \bar{i}
\end{cases}
\end{align*}
\]
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Factorial amplifier:

- Simulate counters bounded by $B!$ with counters bounded by $B$. 

VFAS reachability problem:

- $F_3$ lower bound.
- $d$-EXPSPACE lower bound in dim $d+13$. 

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Reachability in VAS is not Elementary
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- Up to 15 months postdoc position in Bordeaux and/or Cachan (France)