On the 4-color theorem for signed graphs

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Signed graphs : definition

A signed graph (G, σ) is a pair where G is the *underlying graph* and

$$\sigma : \quad E(G) \quad \longrightarrow \quad \{-1, 1\}$$

is called a *signature*.

Signed graphs : switching a vertex

Switching at a vertex v is switching the sign of the incident edges :
$s_v((G, \sigma)) = (G, \sigma')$ where :

$$\sigma'(e) = \begin{cases} -\sigma(e) & \text{if } e \text{ is incident to } v, \\ \sigma(e) & \text{otherwise.} \end{cases}$$

Signed graphs : coloring

Zaslavsky (1982) ; Máčajová, Raspaud and Škoviera (2016) :

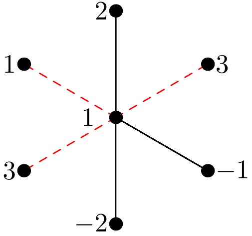
a signed k-coloring :

$$c : \quad V(G) \longrightarrow \begin{array}{l} \{-k/2, ..., -1, 1, ..., k/2\} \text{ if } k \text{ is even} \\ \{-\lfloor k/2 \rfloor, ..., -1, 0, 1, ..., \lfloor k/2 \rfloor\} \text{ if } k \text{ is odd} \end{array}$$

s.t. $c(u) \neq \sigma(uv) \cdot c(v)$.

We denote $\chi(G)$ the minimum k s.t. such a coloring exists.

Signed graphs : coloring and switching



To preserve the coloring when switching at a vertex, it suffices to switch the sign of the color.

Extension of results of proper coloring

Theorem (Máčajová, Raspaud, Škoviera, 2016)
Let G be a simple connected signed graph. If G is not the ***balanced complete graph****, an* ***balanced odd cycle*** *or an* ***unbalanced even cycle****, then $\chi(G) \leq \Delta$.*

- A graph is **balanced** if it is equivalent to a signed graph with **only positive** edges.
- Otherwise it is **unbalanced**.
- Equivalently a graph is **balanced** iff **all its cycles** are balanced.

Extension of results for planar graphs

Theorem (Máčajová, Raspaud, Škoviera, 2016)

Let G be a planar signed graph, then $\chi(G) \leq 5$.

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Let G be a planar signed graph, then :

- *If G is triangle-free, then $\chi(G) \leq 4$.*
- *If G has girth at least 5, then $\chi(G) \leq 3$.*

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Theorem (Jin, Kang, Steffen, 2016)

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Signed graphs : 4-CT for signed graphs ?

Conjecture (Máčajová, Raspaud, Škoviera, 2016)

Every planar signed graph is 4-signed-colorable.

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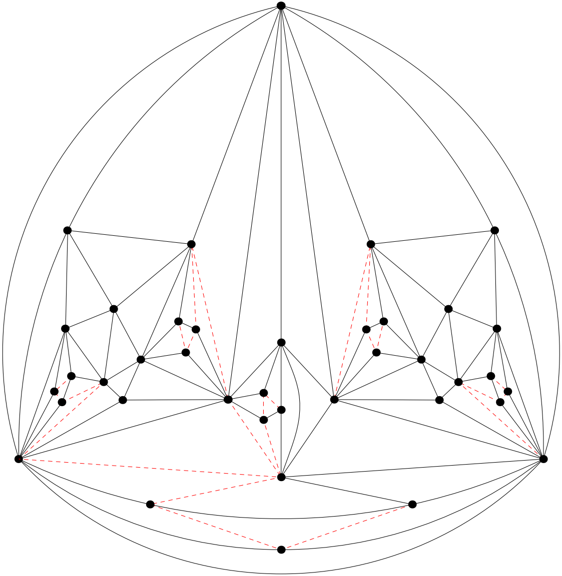
Theorem (Kardoš, N., 2019+)

There exists a signed planar graph on 39 vertices that is not 4-signed-colorable.

A 39-vertex non-4-signed-colorable graph



From 4-colorings to consistent 2-factors

Every planar graph is **4-signed-colorable**.

$\Longleftrightarrow$

Every planar triangulation is **4-signed-colorable**.

$\Longleftrightarrow$

Every cubic 3-connected planar graph with an even number of negative vertices has a **weak edge labeling**.

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Every cubic 3-connected planar graph with an even number of negative vertices has a **strong edge labeling**.

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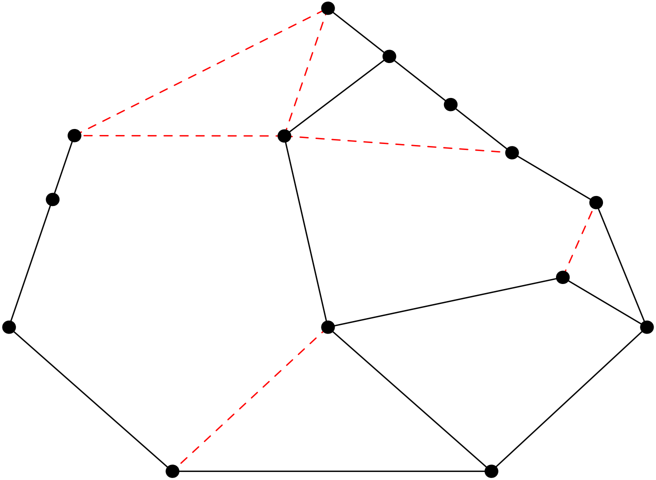
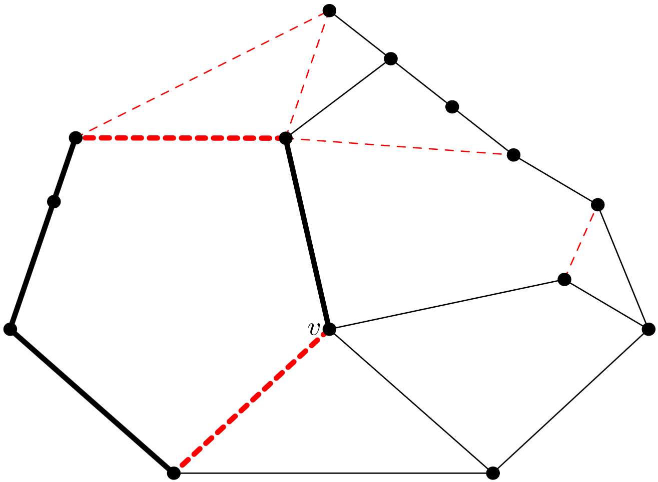
Dual graph of a signed planar graph



Signed graphs : sign of the cycles



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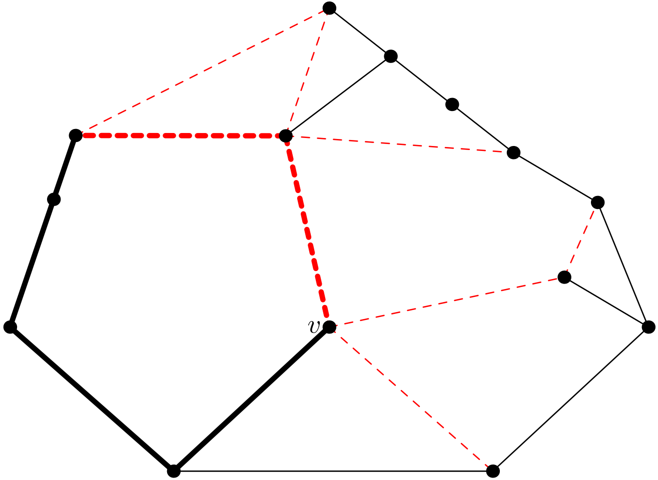
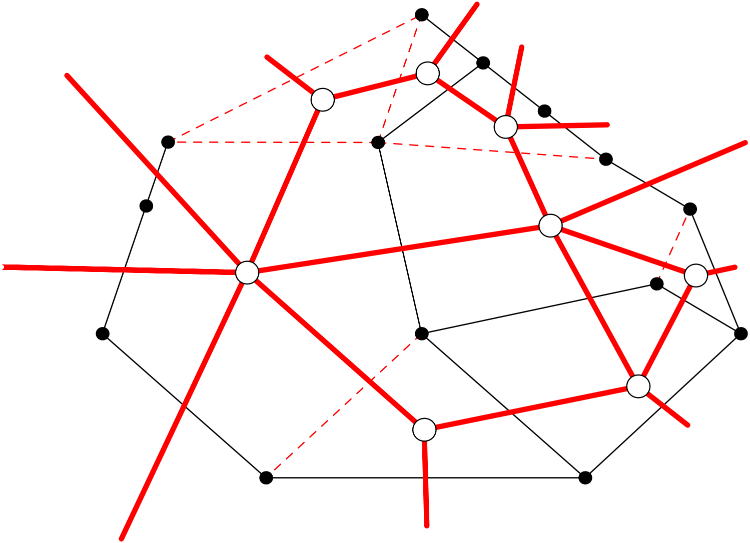


The sign of cycles is preserved when switching.

Dual graph of a signed planar graph



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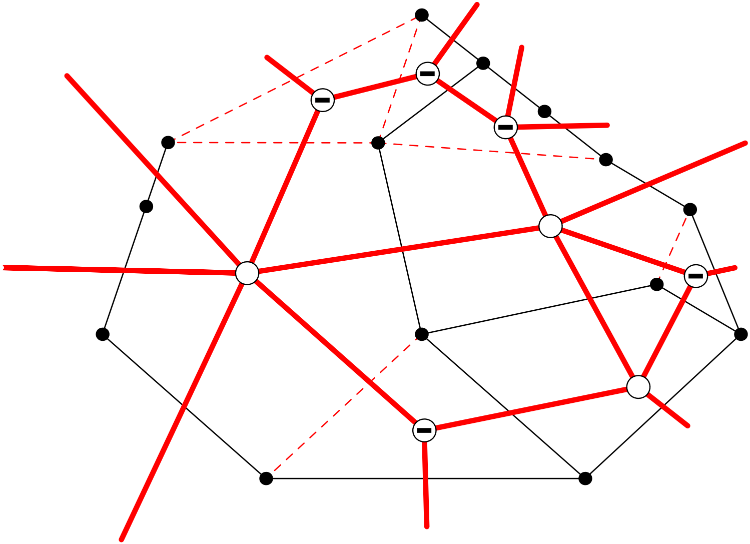


Equivalent signed graphs have the same dual.

Weak signed labeling

A **weak signed edge-labeling** φ^* of G^* is a $\{0, a, b\}$-labeling of the edges of G^* s.t. for each v^* in $V(G^*)$:

Parity invariants :

Positive vertex : • $d_0 \equiv d_a \equiv d_b \equiv d \equiv 1 \pmod 2$

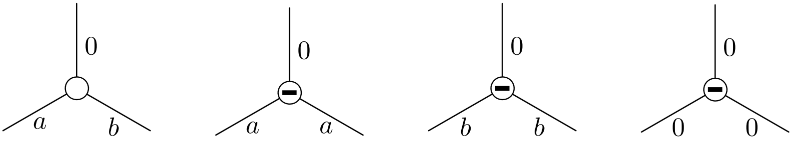
Negative vertex : • $d_0 \equiv d \equiv 1 \pmod 2$
• $d_a \equiv d_b \equiv d + 1 \equiv 0 \pmod 2$

Weak signed labeling of signed triangulations

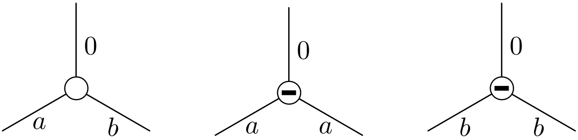
Theorem (Kardoš, N., 2019+)

Let G be a signed planar graph. Then G has a 4-signed coloring iff G^* has a weak edge-labeling.

For the dual of a triangulation, which is cubic, there are only 4 cases :



Strong edge labeling : case of triangulations



- The edges labeled *a* and *b* induce a 2-factor of the graph.
- Only the positive vertices switch the labels.
- Each cycle in the 2-factor must have an even number of positive vertices.

Consistent 2-factor

Let G^* be a 3-connected cubic planar graph with an even number of negative vertices. A 2-factor of G^* is consistent if each cycle in the 2-factor is incident to an **even number of positive vertices**.

Corollary

Let G^* be a 3-connected cubic planar graph with an even number of negative vertices, G^* has a strong signed edge-labeling iff G^* has a consistent 2-factor.

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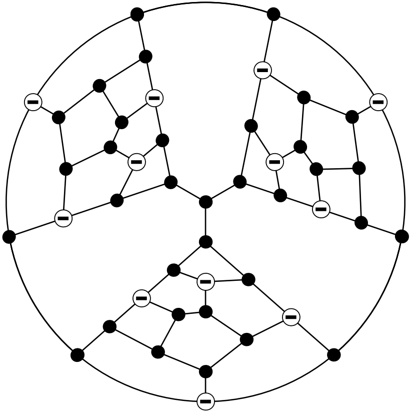
If G^* is **hamiltonian**, then it has a consistent 2-factor.

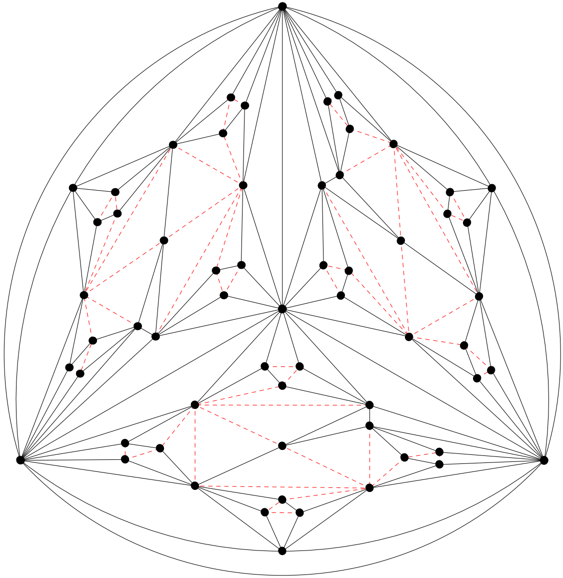
A cubic graph with no consistent 2-factor

Theorem

The Tutte's graph with a choice of negative vertices as depicted in the following figure does not admit a consistent 2-factor.



A graph with no 4-signed-coloring



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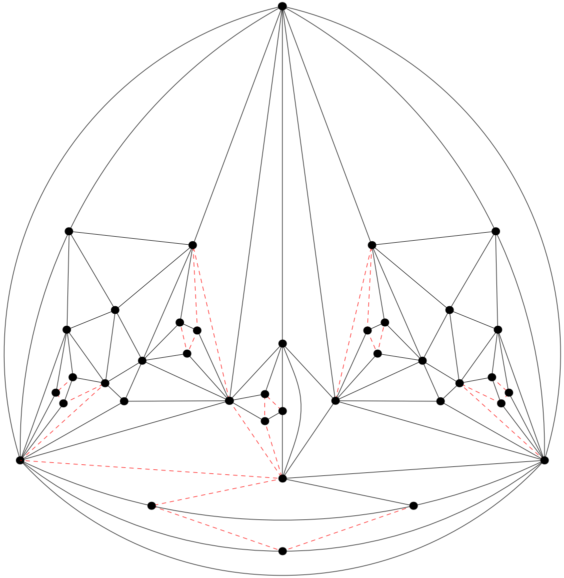
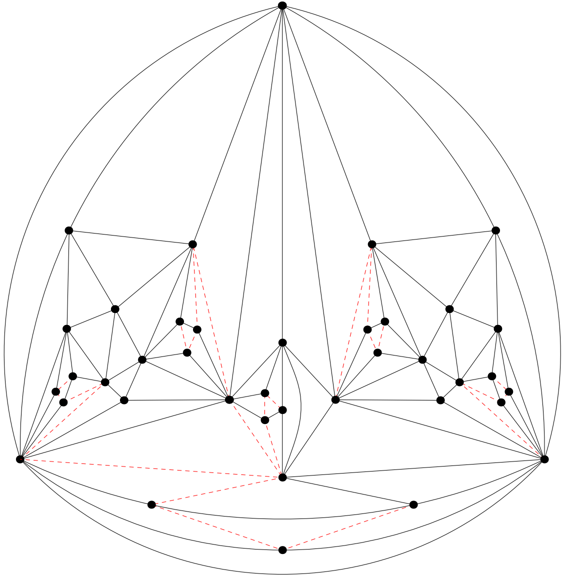


Future work

- Search for a minimum counter-example ($21 \leq n \leq 39$).
- Study the complexity of deciding if a signed planar is 4-colorable.
- Try to translate other types of coloring to edge labeling of the dual.



Thank you !