Decidability of the Domino Problem Under Horizontal Constraints

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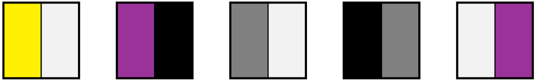
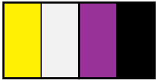
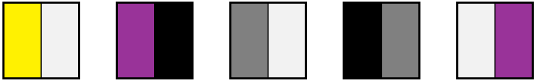
Joint work with Nathalie Aubrun and Mathieu Sablik

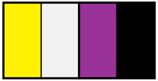
Institut de Mathématiques de Toulouse

EJCIM
June 8^{th}2020

$\mathcal{A} = \{$  $\}$

$\mathcal{F} = \{$  $\dots\}$

$\mathcal{A} = \{$  $\}$

$\mathcal{F} = \{$  $\dots \}$

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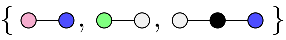
$\mathcal{F} =$ 

$$X_{\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}} \mid \text{ there is no element of } \mathcal{F} \text{ in } x\ \}$$

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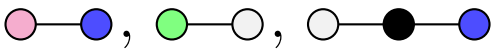
$\mathcal{A} = \{$  $\}$ $\mathcal{F} = \{$  $\}$

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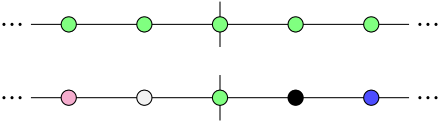

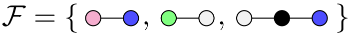

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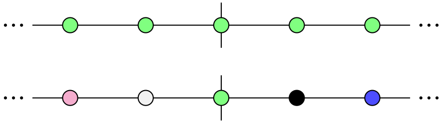

The Domino Problem on $\mathbb{Z}$:

Input: $\mathcal{A}$, $\mathcal{F}$.
Output: *YES* if $X_{\mathcal{F}} \neq \varnothing$, *NO* otherwise.

Theorem (folklore):

$DP(\mathbb{Z})$ is decidable.

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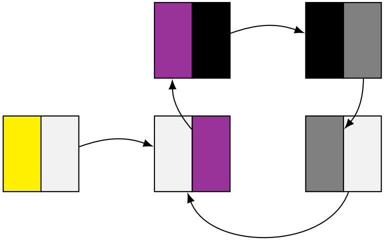

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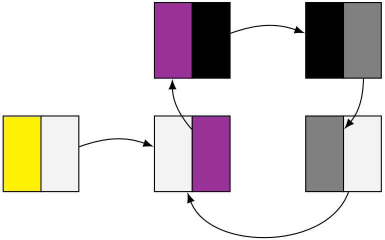
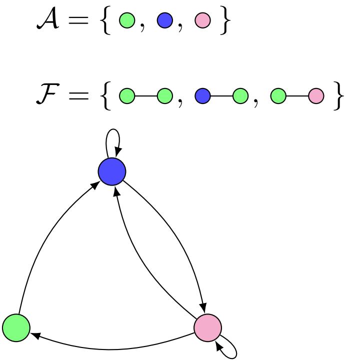
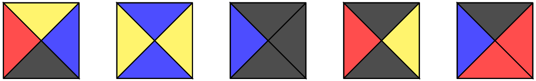
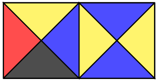
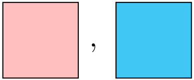
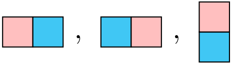
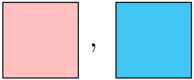
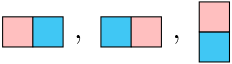




$\mathcal{A} = \{$  $\}$

$\mathcal{F} = \{$  $\dots \}$

$\mathcal{A} = \{$  $\}$ $\mathcal{F} = \{$  $\}$

$\mathcal{A} = \{$  $\}$ $\mathcal{F} = \{$  $\}$

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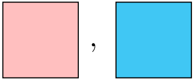
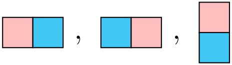
$\mathcal{A} = \{$  $\}$ $\mathcal{F} = \{$  $\}$

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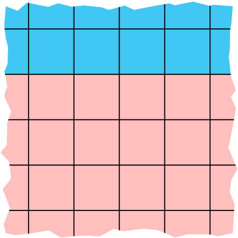
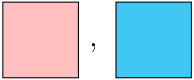
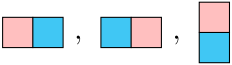


$\mathcal{A} = \{$  $\}$

$\mathcal{F} = \{$  $\}$

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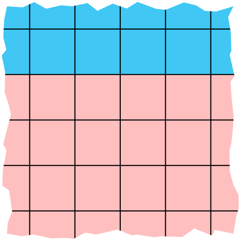

The Domino Problem on $\mathbb{Z}^2$:

Input: $\mathcal{A}$, $\mathcal{F}$.

Output: *YES* if $X_{\mathcal{F}} \neq \varnothing$, *NO* otherwise.

Theorem (Berger 66, Robinson 71, Mozes 89, Kari 96...):

$DP(\mathbb{Z}^2)$ is undecidable.

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Reduction to the Halting Problem.

$X_{H,V}$:

H and V SFTs on $\mathbb{Z}$ with alphabet $\mathcal{A}$.

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$X_{H,V}$ is the SFT on $\mathbb{Z}^2$ using $\mathcal{A}$ with lines in H and columns in V.

$$\mathcal{F}(H) = \{ \bullet\!-\!\bullet, \circ\!-\!\circ \}$$

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$\mathcal{F}(H) = \{$ ●—●, ○—○ $\}$

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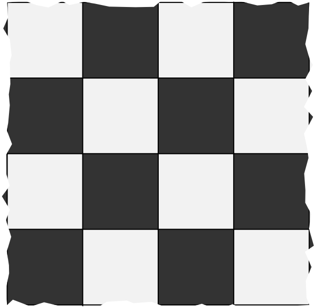
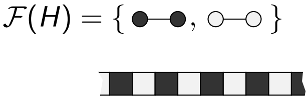
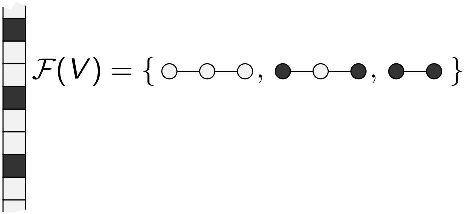
Compatible




F(H) =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Incompatible

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H an SFT on $\mathbb{Z}$.

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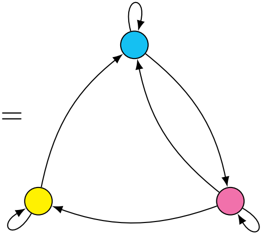
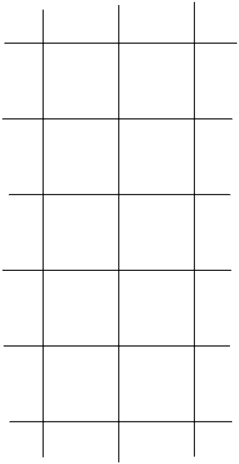
Remark:

The answer depends on H!

$H =$ 



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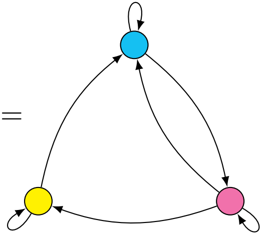

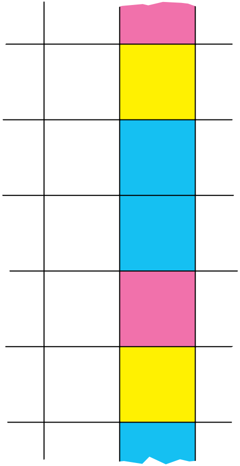

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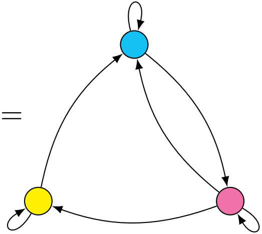
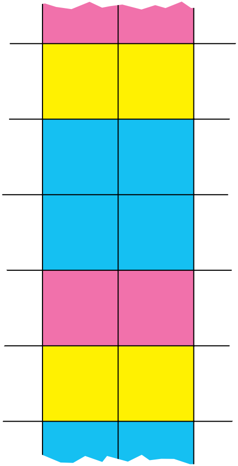
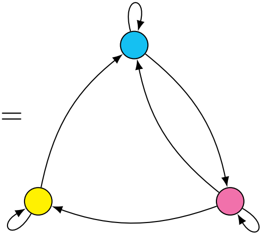
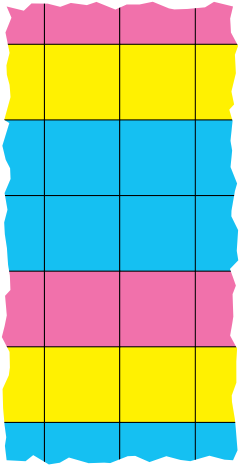
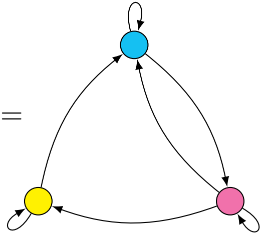
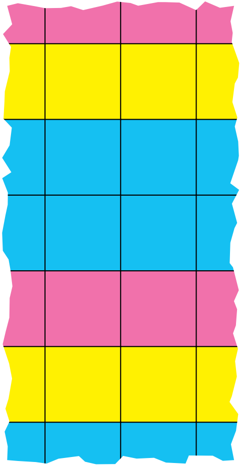




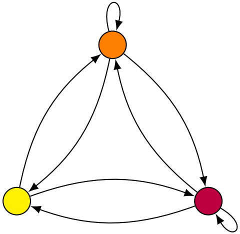
$H =$ 

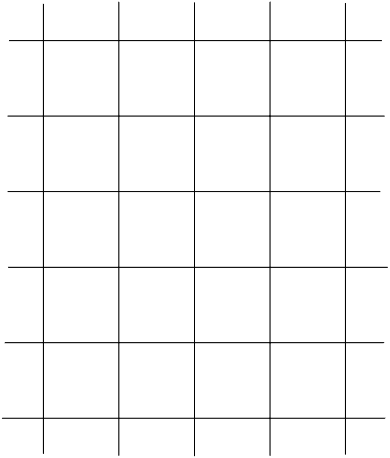


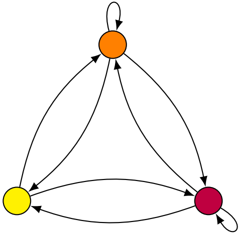
$H =$ 

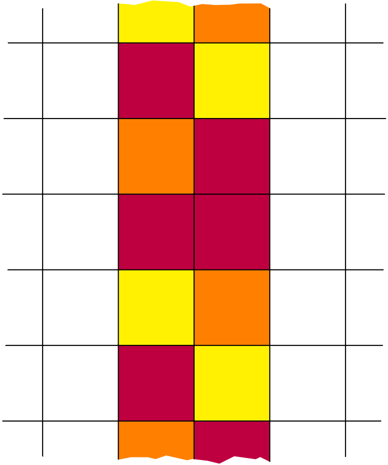


DP_H is decidable with this H.

$H =$ 



$H =$ 



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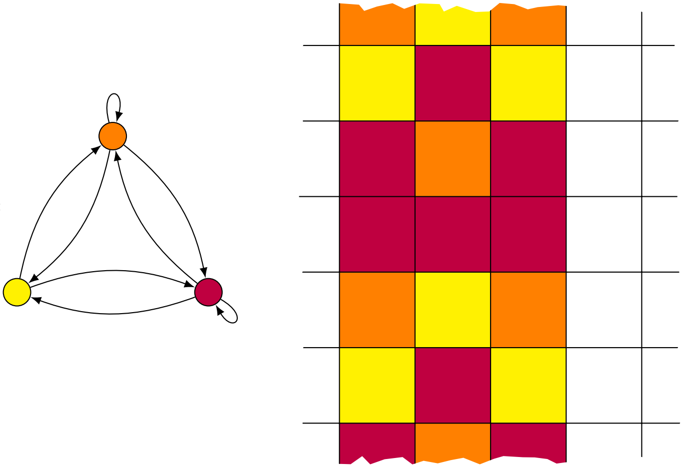
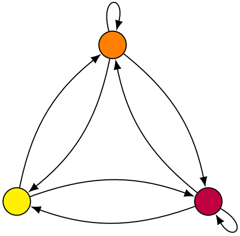
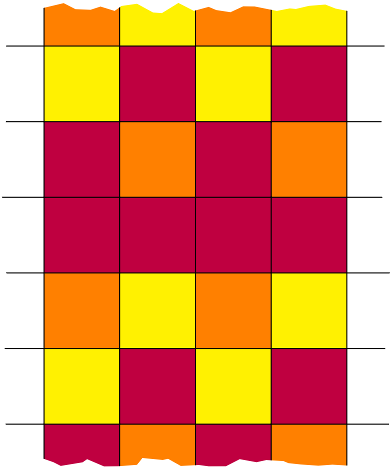
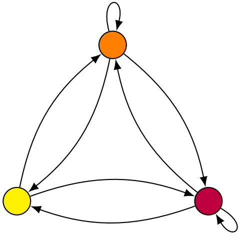
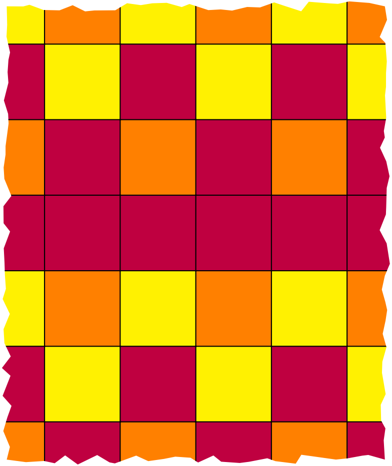
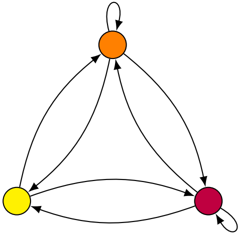


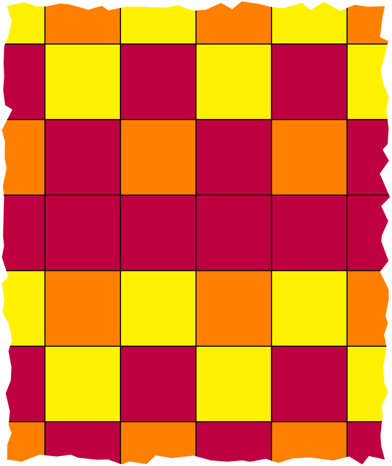
$H =$ 



$H =$ 



$H =$ 



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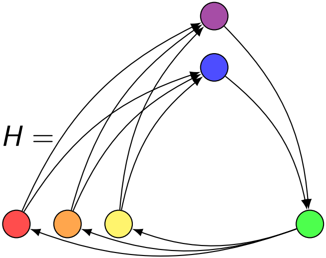
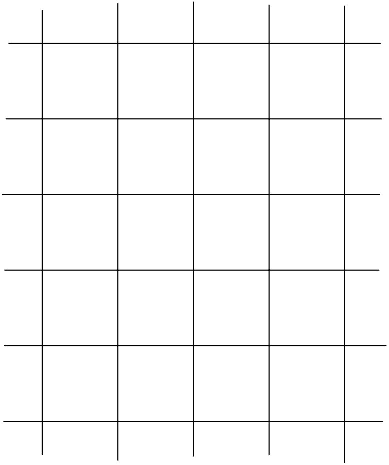
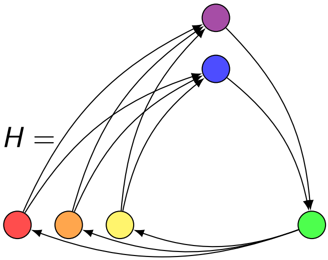
$H =$ 

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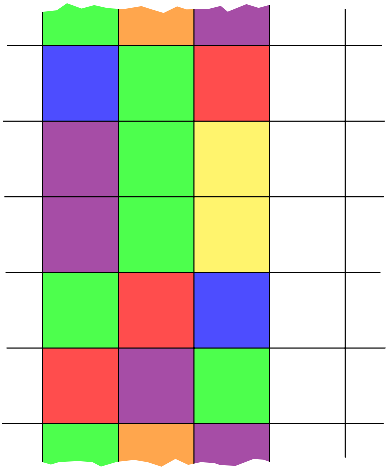
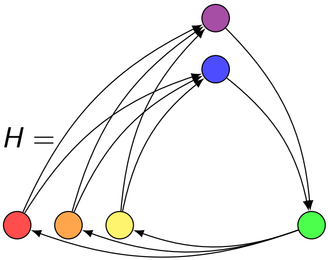
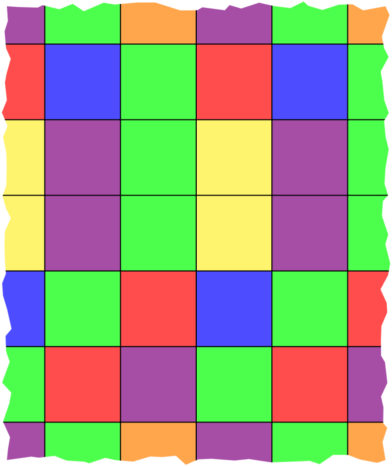


$H =$ 



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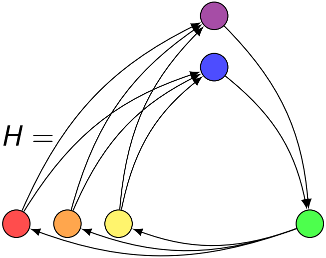

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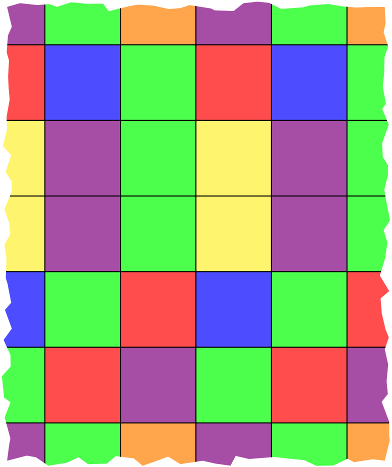
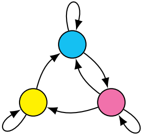
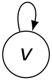
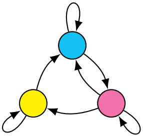


 $\Longrightarrow \forall v,$ 

$\Longrightarrow$ reflexive type



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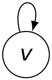

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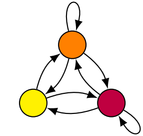
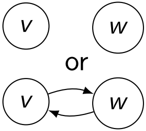
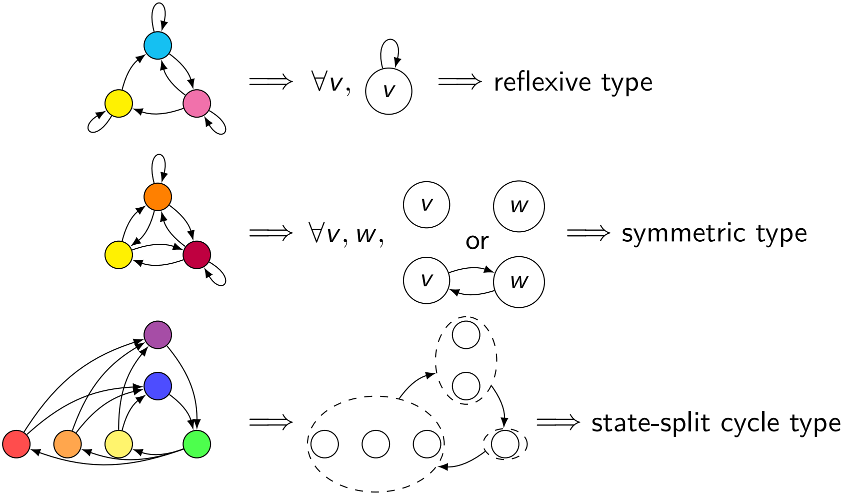


$\Longrightarrow \forall v, w,$



$\Longrightarrow$ symmetric type


⟹ ∀v, v ⟹ reflexive type
⟹ ∀v, w, v w or v w ⟹ symmetric type
⟹ state-split cycle type

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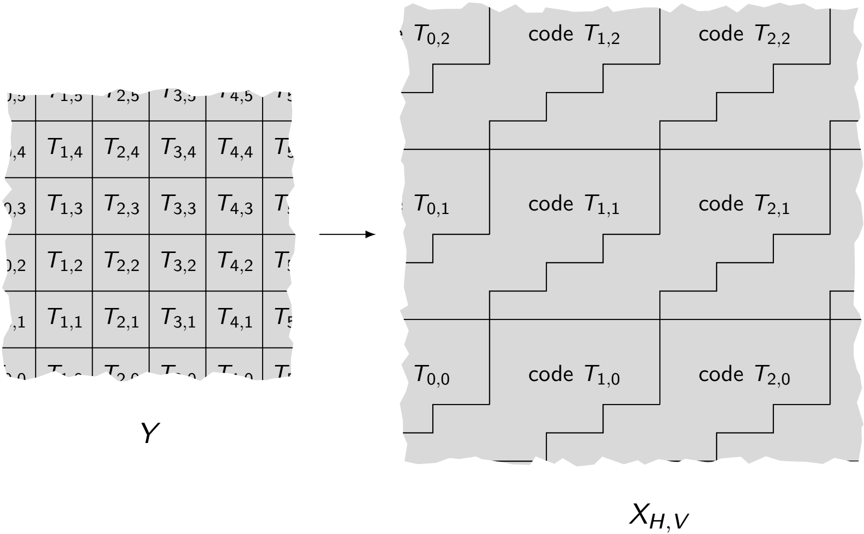
We show that for any Y SFT on $\mathbb{Z}^2$, there are vertical rules V_Y so that X_{H,V_Y} reproduces the configurations of Y.



$T_{1,5}$ $T_{2,5}$ $T_{3,5}$ $T_{4,5}$
$T_{1,4}$ $T_{2,4}$ $T_{3,4}$ $T_{4,4}$
$T_{1,3}$ $T_{2,3}$ $T_{3,3}$ $T_{4,3}$
$T_{1,2}$ $T_{2,2}$ $T_{3,2}$ $T_{4,2}$
$T_{1,1}$ $T_{2,1}$ $T_{3,1}$ $T_{4,1}$
Y
$T_{0,2}$ code $T_{1,2}$ code $T_{2,2}$
$T_{0,1}$ code $T_{1,1}$ code $T_{2,1}$
$T_{0,0}$ code $T_{1,0}$ code $T_{2,0}$
$X_{H,V}$

Proposition (Aubrun-E.-Sablik, 2020):

For any Y SFT on $\mathbb{Z}^2$,
for any "complicated enough" H with length-2 constraints,
there is some V_Y so that X_{H,V_Y} reproduces the configurations of Y.

Thank you for your attention!

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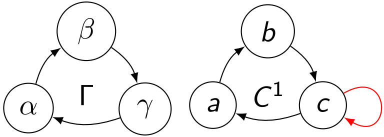
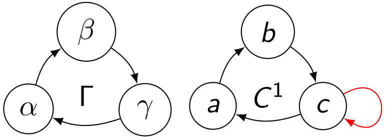
- higher dimension with H on $\mathbb{Z}^k$, input V on $\mathbb{Z}^{d-k}$, $X_{H,V}$ on $\mathbb{Z}^d$.

$H =$ 

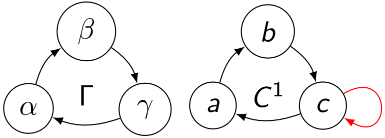
$Y = \{\tau_1, \tau_2, \tau_3\}$

$H =$



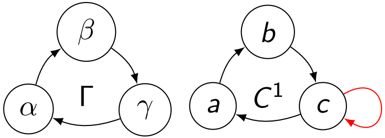
$Y = \{\tau_1, \tau_2, \tau_3\}$

β	γ	α	β	γ	α

$H =$ 

$$Y = \{\tau_1, \tau_2, \tau_3\}$$

β	γ	α	β	γ	α
b	c	a	b	c	a
b	c	a	b	c	a
b	c	a	b	c	a

$H =$ 

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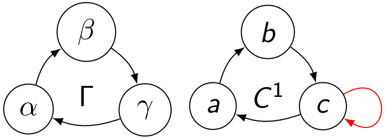
β	γ	α	β	γ	α
b	c	c	c	c	a
b	c	a	b	c	c
c	c	a	b	c	a

$H =$ 

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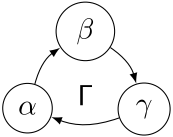
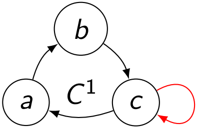
β	γ	α	β	γ	α
b	c	c	c	c	a
b	c	a	b	c	c
c	c	a	b	c	a
c	c	c	c	a	b
c	a	b	c	c	c
c	a	b	c	a	b

$H =$ 



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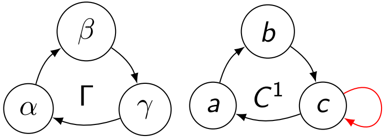
β	γ	α	β	γ	α
b	c	c	c	c	a
b	c	a	b	c	c
c	c	a	b	c	a
c	c	c	c	a	b
c	a	b	c	c	c
c	a	b	c	a	b
c	c	c	a	b	c
a	b	c	c	c	c
a	b	c	a	b	c

$H =$ 

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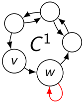
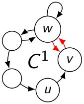
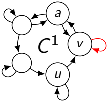
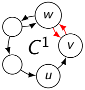
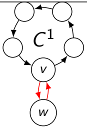
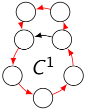
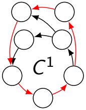
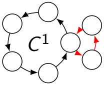
β	γ	α	β	γ	α
b	c	c	c	c	a
b	c	a	b	c	c
c	c	a	b	c	a
c	c	c	c	a	b
c	a	b	c	c	c
c	a	b	c	a	b
c	c	c	a	b	c
a	b	c	c	c	c
a	b	c	a	b	c

Table of the main cases, each of them illustrated with an example.

Loops			No loop				
			Bidirectional edges		No bidirectional edge		
							
Case 1.1	Case 1.2	Case 1.3	Case 2.1	Case 2.2	Case 3.1	Case 3.2	Case 3.3