

# Decidability of the Domino Problem Under Horizontal Constraints

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EJCIM

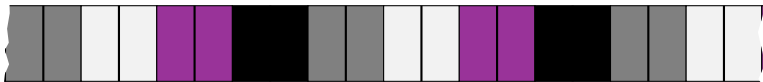
June 8<sup>th</sup>2020

$$\mathcal{A} = \left\{ \begin{array}{|c|c|} \hline \text{yellow} & \text{white} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \text{purple} & \text{black} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \text{gray} & \text{white} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \text{black} & \text{gray} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \text{white} & \text{purple} \\ \hline \end{array} \right\}$$

$$\mathcal{F} = \left\{ \begin{array}{|c|c|c|c|} \hline \text{yellow} & \text{white} & \text{purple} & \text{black} \\ \hline \end{array} \dots \right\}$$

$$\mathcal{A} = \left\{ \begin{array}{|c|c|} \hline \text{yellow} & \text{white} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \text{purple} & \text{black} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \text{gray} & \text{white} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \text{black} & \text{gray} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \text{white} & \text{purple} \\ \hline \end{array} \right\}$$

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$$\mathcal{A} = \{ \text{green circle}, \text{white circle}, \text{black circle}, \text{pink circle}, \text{blue circle} \}$$

$$\mathcal{F} = \{ \text{pink circle} - \text{blue circle}, \text{green circle} - \text{white circle}, \text{white circle} - \text{black circle} - \text{blue circle} \}$$

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$$X_{\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}} \mid \text{there is no element of } \mathcal{F} \text{ in } x \}$$

Called a **subshift**.

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Called a **subshift of finite type (SFT)** if finite number of constraints.

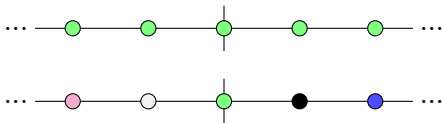
$$\mathcal{A} = \{ \text{●}, \text{○}, \text{●}, \text{○}, \text{●} \}$$

$$\mathcal{F} = \{ \text{○—●}, \text{●—○}, \text{○—●—○} \}$$

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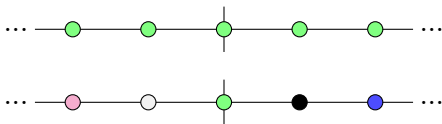
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## The Domino Problem on $\mathbb{Z}$ :

Input:  $\mathcal{A}, \mathcal{F}$ .

Output: YES if  $X_{\mathcal{F}} \neq \emptyset$ , NO otherwise.

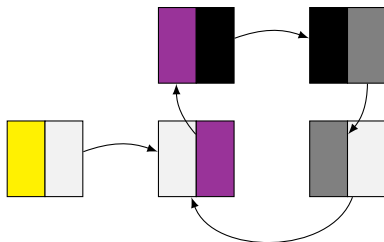


Theorem (folklore):

$DP(\mathbb{Z})$  is decidable.

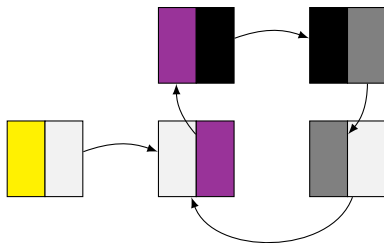
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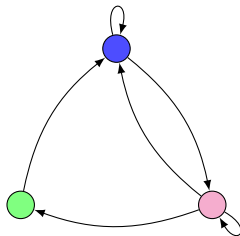
# Theorem (folklore):

$DP(\mathbb{Z})$  is decidable.



$$\mathcal{A} = \{ \text{green}, \text{blue}, \text{pink} \}$$

$$\mathcal{F} = \{ \text{green-green}, \text{blue-green}, \text{green-pink} \}$$



$$\mathcal{A} = \left\{ \begin{array}{c} \text{[Red, Yellow, Blue, Gray square]} \\ \text{[Yellow, Blue, Gray square]} \\ \text{[Blue, Gray square]} \\ \text{[Red, Yellow, Gray square]} \\ \text{[Blue, Red, Gray square]} \end{array} \right\}$$

$$\mathcal{F} = \left\{ \begin{array}{c} \text{[Red, Yellow, Blue, Gray square]} \\ \text{[Yellow, Blue, Gray square]} \end{array} \dots \right\}$$

$$\mathcal{A} = \left\{ \begin{array}{c} \text{Red Square} \\ \text{Blue Square} \end{array} \right\}$$

$$\mathcal{F} = \left\{ \begin{array}{c} \text{Red-Blue} \\ \text{Blue-Red} \\ \text{Red-Blue} \\ \text{Blue-Red} \end{array} \right\}$$

$$\mathcal{A} = \left\{ \begin{array}{c} \square \\ \square \end{array}, \begin{array}{c} \square \\ \square \end{array} \right\}$$

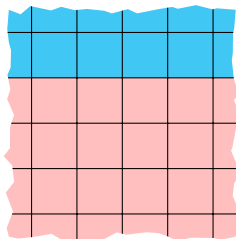
$$\mathcal{F} = \left\{ \begin{array}{cc} \square & \square \\ \square & \square \end{array}, \begin{array}{cc} \square & \square \\ \square & \square \end{array}, \begin{array}{c} \square \\ \square \end{array} \right\}$$

$$X_{\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}^2} \mid \text{there is no element of } \mathcal{F} \text{ in } x \}$$

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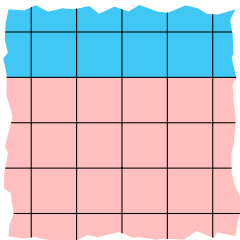
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$$X_{\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}^2} \mid \text{there is no element of } \mathcal{F} \text{ in } x\}$$



## The Domino Problem on $\mathbb{Z}^2$ :

Input:  $\mathcal{A}, \mathcal{F}$ .

Output: YES if  $X_{\mathcal{F}} \neq \emptyset$ , NO otherwise.



Theorem (Berger 66, Robinson 71, Mozes 89, Kari 96...):

$DP(\mathbb{Z}^2)$  is undecidable.

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Reduction to the Halting Problem.

$X_{H,V}$ : $H$  and  $V$  SFTs on  $\mathbb{Z}$  with alphabet  $\mathcal{A}$ .

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$$\mathcal{F}(H) = \{ \bullet\text{---}\bullet, \circ\text{---}\circ \}$$



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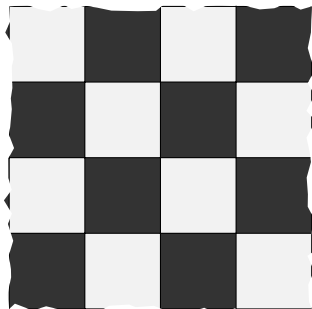
$$\mathcal{F}(V) = \{ \bullet\text{---}\bullet, \circ\text{---}\circ \}$$

$$\mathcal{F}(H) = \{ \bullet\text{---}\bullet, \circ\text{---}\circ \}$$



$$\mathcal{F}(V) = \{ \bullet\text{---}\bullet, \circ\text{---}\circ \}$$

Compatible



$$\mathcal{F}(H) = \{\bullet\text{---}\bullet, \circ\text{---}\circ\}$$



$$\mathcal{F}(V) = \{\circ\text{---}\circ\text{---}\circ, \bullet\text{---}\circ\text{---}\bullet, \bullet\text{---}\bullet\}$$



$$\mathcal{F}(H) = \{ \bullet\text{---}\bullet, \circ\text{---}\circ \}$$



$$\mathcal{F}(V) = \{ \circ\text{---}\circ\text{---}\circ, \bullet\text{---}\circ\text{---}\bullet, \bullet\text{---}\bullet \}$$



Incompatible

$DP_H$ : $H$  an SFT on  $\mathbb{Z}$ .

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$DP_H$ :

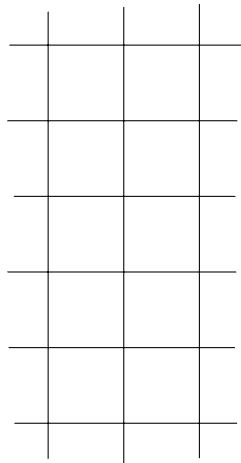
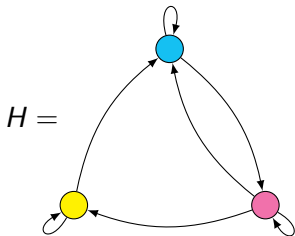
$H$  an SFT on  $\mathbb{Z}$ .

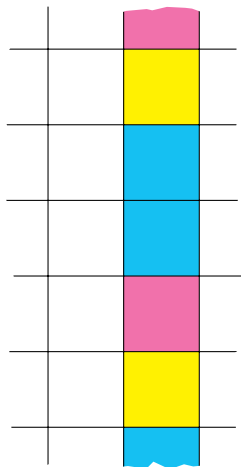
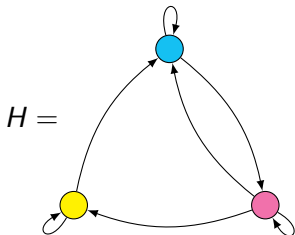
Input: set of vertical forbidden patterns  $\mathcal{F}_V$ .

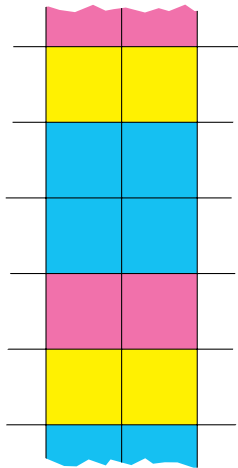
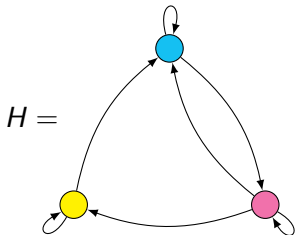
Output: YES if  $X_{H,V} \neq \emptyset$ , NO otherwise.

## Remark:

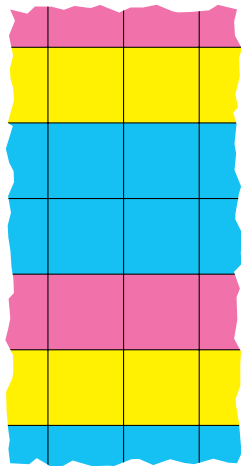
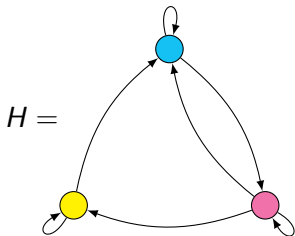
The answer depends on  $H$ !

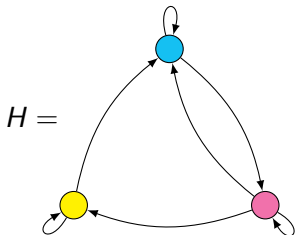




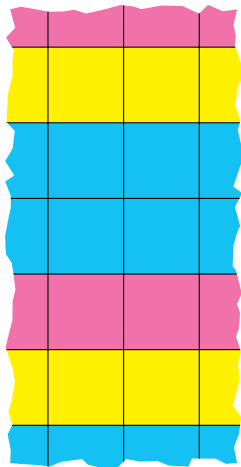




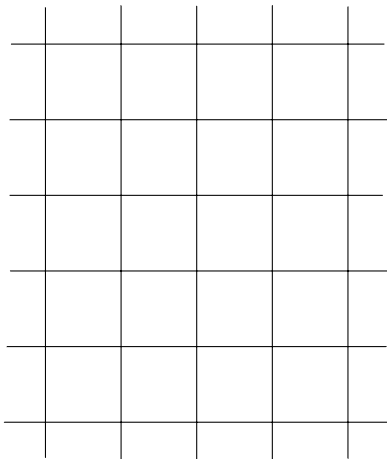
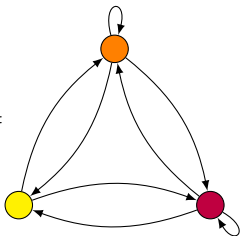




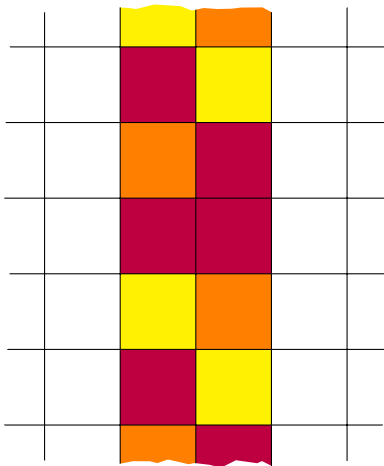
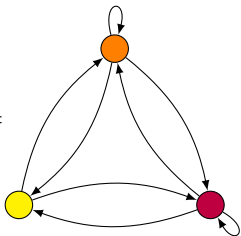
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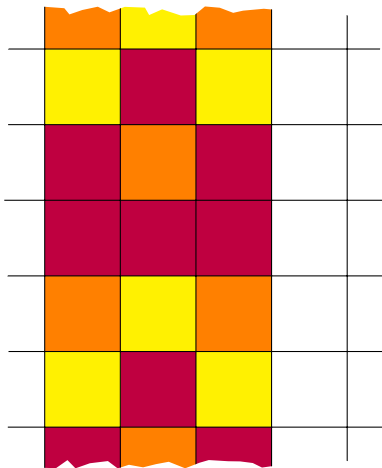
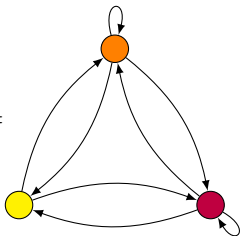
$H =$



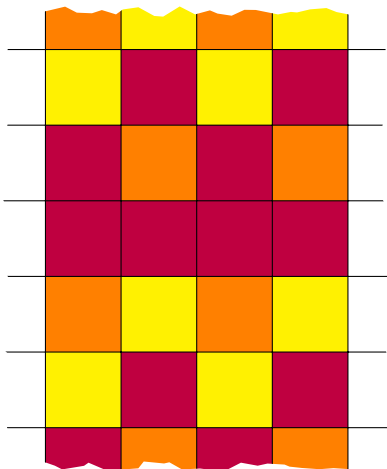
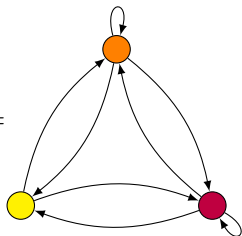
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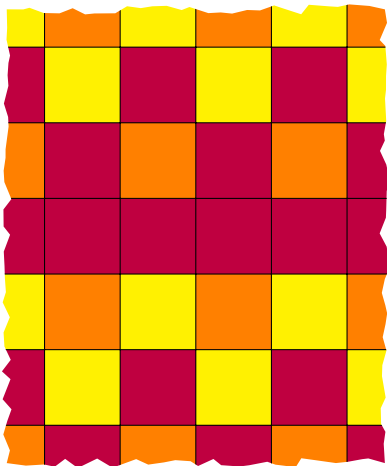
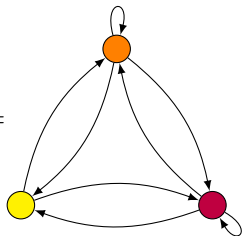
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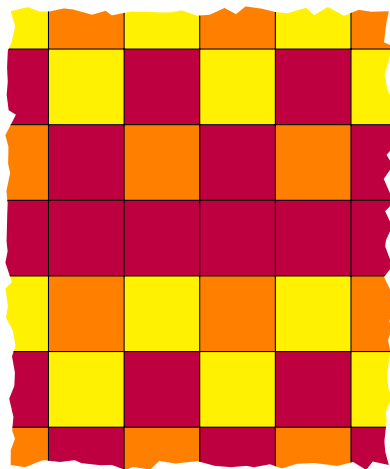
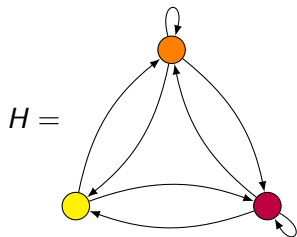


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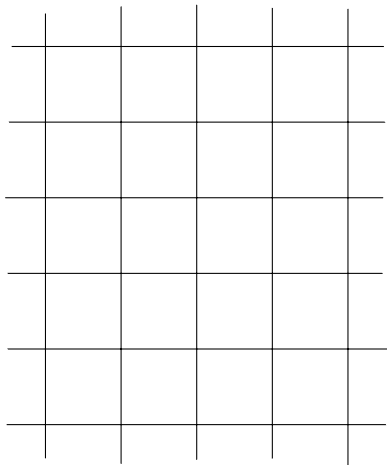
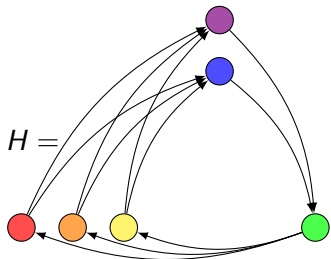
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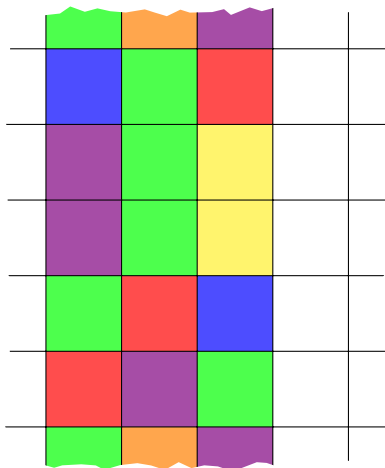
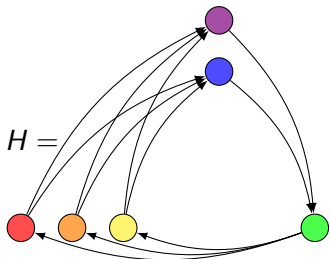


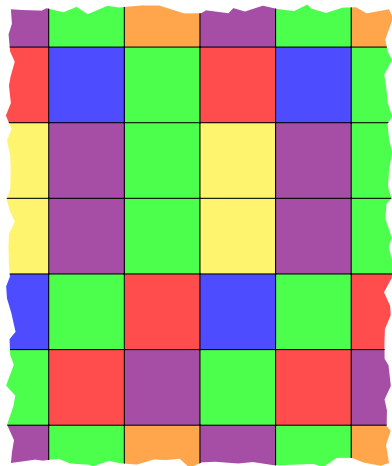
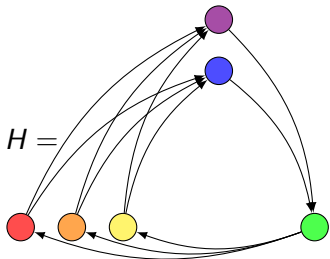


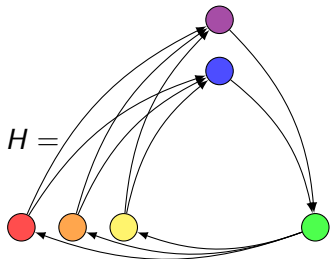
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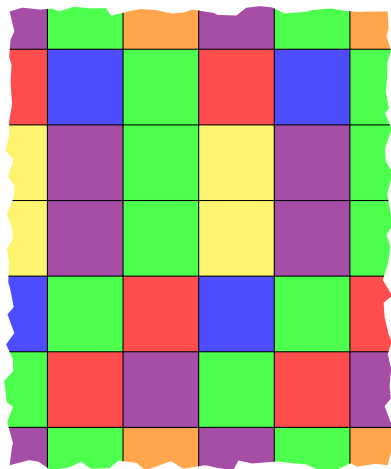


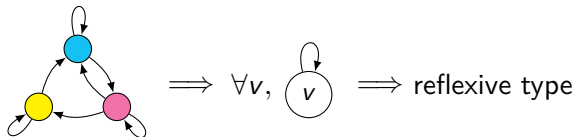


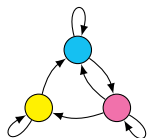




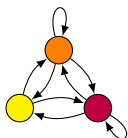
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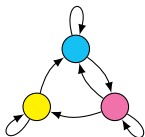




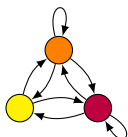
$\Rightarrow \forall v, \text{ (node with self-loop)} \Rightarrow$  reflexive type



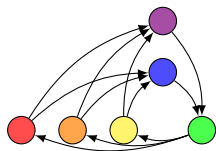
$\Rightarrow \forall v, w, \text{ (nodes v and w)} \text{ or } \text{ (nodes v and w with edge)} \Rightarrow$  symmetric type



$$\Rightarrow \forall v, \text{ (node } v \text{ with self-loop)} \Rightarrow \text{reflexive type}$$



$$\Rightarrow \forall v, w, \text{ (nodes } v \text{ and } w \text{ with bidirectional arrows)} \Rightarrow \text{symmetric type}$$



$$\Rightarrow \text{ (graph with dashed loops) } \Rightarrow \text{state-split cycle type}$$

## Question:

These are examples with easy decidability. What about other  $H$ s?



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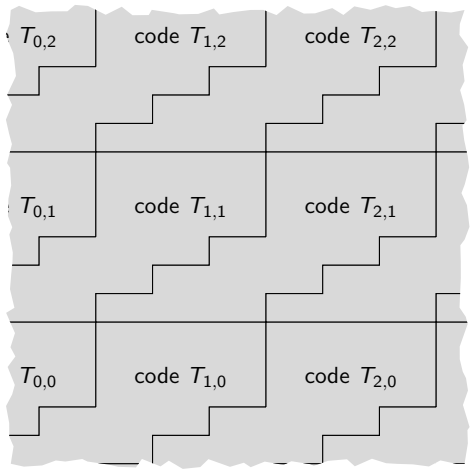
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We reduce to...  $DP(\mathbb{Z}^2)$ .

We show that for any  $Y$  SFT on  $\mathbb{Z}^2$ , there are vertical rules  $V_Y$  so that  $X_{H, V_Y}$  reproduces the configurations of  $Y$ .

$T_{0,5}$	$T_{1,5}$	$T_{2,5}$	$T_{3,5}$	$T_{4,5}$	$T_{5,5}$
$T_{0,4}$	$T_{1,4}$	$T_{2,4}$	$T_{3,4}$	$T_{4,4}$	$T_{5,4}$
$T_{0,3}$	$T_{1,3}$	$T_{2,3}$	$T_{3,3}$	$T_{4,3}$	$T_{5,3}$
$T_{0,2}$	$T_{1,2}$	$T_{2,2}$	$T_{3,2}$	$T_{4,2}$	$T_{5,2}$
$T_{0,1}$	$T_{1,1}$	$T_{2,1}$	$T_{3,1}$	$T_{4,1}$	$T_{5,1}$
$T_{0,0}$	$T_{1,0}$	$T_{2,0}$	$T_{3,0}$	$T_{4,0}$	$T_{5,0}$

$Y$



$X_{H,V}$

## Proposition (Aubrun-E.-Sablik, 2020):

For any  $Y$  SFT on  $\mathbb{Z}^2$ ,  
for any “complicated enough”  $H$  with length-2 constraints,  
there is some  $V_Y$  so that  $X_{H, V_Y}$  reproduces the configurations of  $Y$ .

Thank you for your attention!



Possible extensions:

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- $H$  with more complex constraints

$$\mathcal{A} = \{a, b\}$$

$$\mathcal{F} = \{aba, bab, bbb\}$$

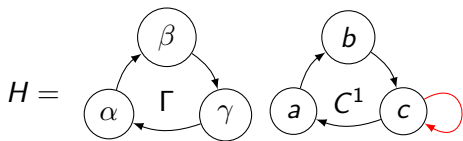
Possible extensions:

- $H$  with more complex constraints

$$\mathcal{A} = \{a, b\}$$

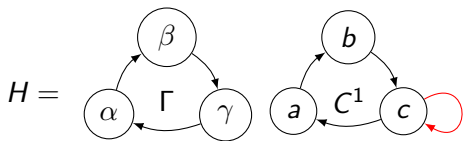
$$\mathcal{F} = \{aba, bab, bbb\}$$

- higher dimension with  $H$  on  $\mathbb{Z}^k$ , input  $V$  on  $\mathbb{Z}^{d-k}$ ,  $X_{H,V}$  on  $\mathbb{Z}^d$ .



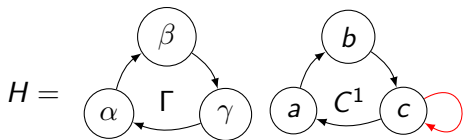
$$Y = \{\tau_1, \tau_2, \tau_3\}$$





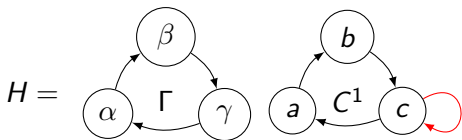
$$Y = \{\tau_1, \tau_2, \tau_3\}$$

$\beta$	$\gamma$	$\alpha$	$\beta$	$\gamma$	$\alpha$
$b$	$c$	$a$	$b$	$c$	$a$
$b$	$c$	$a$	$b$	$c$	$a$
$b$	$c$	$a$	$b$	$c$	$a$



$$Y = \{\tau_1, \tau_2, \tau_3\}$$

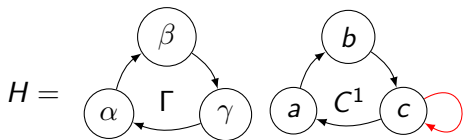
$\beta$	$\gamma$	$\alpha$	$\beta$	$\gamma$	$\alpha$
$b$	$c$	$c$	$c$	$c$	$a$
$b$	$c$	$a$	$b$	$c$	$c$
$c$	$c$	$a$	$b$	$c$	$a$



$Y = \{\tau_1, \tau_2, \tau_3\}$

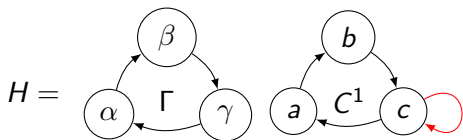
$\beta$	$\gamma$	$\alpha$	$\beta$	$\gamma$	$\alpha$
$b$	$c$	$c$	$c$	$c$	$a$
$b$	$c$	$a$	$b$	$c$	$c$
$c$	$c$	$a$	$b$	$c$	$a$
$c$	$c$	$c$	$c$	$a$	$b$
$c$	$a$	$b$	$c$	$c$	$c$
$c$	$a$	$b$	$c$	$a$	$b$





$Y = \{\tau_1, \tau_2, \tau_3\}$

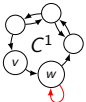
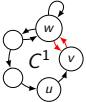
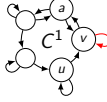

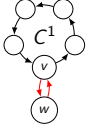
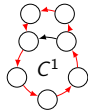
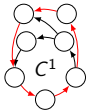
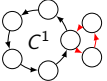
$\beta$	$\gamma$	$\alpha$	$\beta$	$\gamma$	$\alpha$
$b$	$c$	$c$	$c$	$c$	$a$
$b$	$c$	$a$	$b$	$c$	$c$
$c$	$c$	$a$	$b$	$c$	$a$
$c$	$c$	$c$	$c$	$a$	$b$
$c$	$a$	$b$	$c$	$c$	$c$
$c$	$a$	$b$	$c$	$a$	$b$
$c$	$c$	$c$	$a$	$b$	$c$
$a$	$b$	$c$	$c$	$c$	$c$
$a$	$b$	$c$	$a$	$b$	$c$



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$b$	$c$	$a$	$b$	$c$	$c$
$c$	$c$	$a$	$b$	$c$	$a$
$c$	$c$	$c$	$c$	$a$	$b$
$c$	$a$	$b$	$c$	$c$	$c$
$c$	$a$	$b$	$c$	$a$	$b$
$c$	$c$	$c$	$a$	$b$	$c$
$a$	$b$	$c$	$c$	$c$	$c$
$a$	$b$	$c$	$a$	$b$	$c$

Table of the main cases, each of them illustrated with an example.

Loops			No loop				
			Bidirectional edges		No bidirectional edge		
							
Case 1.1	Case 1.2	Case 1.3	Case 2.1	Case 2.2	Case 3.1	Case 3.2	Case 3.3