Decidability of the Domino Problem Under Horizontal Constraints

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\[ \mathcal{A} = \{ \text{ } \} \]

\[ \mathcal{F} = \{ \text{ ... } \} \]
\[ A = \{ \begin{array}{ccc} \text{yellow} & \text{white} & \text{black} \\ \text{purple} & \text{gray} & \text{black} \\ \text{white} & \text{gray} & \text{black} \end{array} \} \]

\[ F = \{ \begin{array}{c} \text{yellow} & \text{white} & \text{purple} \\ \text{black} & \ldots \end{array} \} \]
\[ A = \{ \circ, \circ, \bullet, \circ, \circ \} \quad \text{and} \quad F = \{ \circ \bullet, \circ \circ, \circ \bullet \bullet \circ \} \]
\[\mathcal{A} = \{ \bigcirc, \bullet, \bigcirc, \bigcirc, \bullet \} \quad \mathcal{F} = \{ \bigcirc \bullet, \bigcirc \bigcirc, \bigcirc \bullet \bullet \} \]

\[X_\mathcal{F} = \{ x \in \mathcal{A}^\mathbb{Z} | \text{ there is no element of } \mathcal{F} \text{ in } x \} \]

Called a subshift.
\[ \mathcal{A} = \{ \bullet, \circ, \circ\} \quad \mathcal{F} = \{ \circ\bullet, \circ\circ, \circ\circ\circ\circ\circ\} \]

\[ X_{\mathcal{F}} = \{ x \in \mathcal{A}^\mathbb{Z} \mid \text{there is no element of } \mathcal{F} \text{ in } x \} \]

Called a subshift.
Called a subshift of finite type (SFT) if finite number of constraints.
\[ A = \{ \bullet, \circ, \bullet, \circ, \bullet \} \quad \text{and} \quad F = \{ \circ\bullet, \circ\circ, \circ\bullet\bullet \} \]

\[ X_F = \{ x \in A^\mathbb{Z} \mid \text{there is no element of } F \text{ in } x \} \]

Called a subshift.

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\[ A = \{ \bullet, \circ, \bullet, \circ, \bullet \} \quad \text{and} \quad \mathcal{F} = \{ \circ\bullet, \bullet\circ, \circ\bullet\bullet \} \]

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Called a subshift.
Called a subshift of finite type (SFT) if finite number of constraints.

The Domino Problem on \( \mathbb{Z} \):

Input: \( A, \mathcal{F} \).
Output: YES if \( X_{\mathcal{F}} \neq \emptyset \), NO otherwise.
Theorem (folklore):

$DP(\mathbb{Z})$ is decidable.
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\( DP(\mathbb{Z}) \) is decidable.
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$DP(\mathbb{Z})$ is decidable.

$A = \{ \circ, \bullet, \circ \}$

$\mathcal{F} = \{ \circ \circ , \bullet \bullet , \circ \circ \}$
\[ A = \{ \begin{array}{cccc}
\text{Red} & \text{Blue} & \text{Black} & \text{Yellow} \\
\text{Red} & \text{Blue} & \text{Black} & \text{Yellow} \\
\text{Red} & \text{Blue} & \text{Black} & \text{Yellow} \\
\text{Red} & \text{Blue} & \text{Black} & \text{Yellow} \\
\end{array} \} \]

\[ \mathcal{F} = \{ \begin{array}{c}
\begin{array}{c}
\text{Red} \\
\text{Blue} \\
\text{Black} \\
\text{Yellow}
\end{array}
\end{array} \ldots \} \]
\[ A = \{ \text{ }
\begin{array}{c}
\text{ }
\end{array}
\} \quad F = \{ \text{ }
\begin{array}{c}
\text{ }
\end{array}
\} \]

The Domino Problem on \( \mathbb{Z}^2 \):

Input: \( A, F \).

Output: YES if \( X_F \neq \emptyset \), NO otherwise.
\[ \mathcal{A} = \{ \ \square, \ [ \square ] \} \quad \mathcal{F} = \{ \ [\square \square], \ [\square \square \square], \ [\square \square] \} \]

\[ X_{\mathcal{F}} = \{ x \in \mathcal{A}^{\mathbb{Z}^2} \mid \text{there is no element of } \mathcal{F} \text{ in } x \} \]
\[ A = \{ \begin{array}{} \text{square} \end{array}, \begin{array}{} \text{square} \end{array} \} \quad F = \{ \begin{array}{} \text{rectangle} \end{array}, \begin{array}{} \text{rectangle} \end{array}, \begin{array}{} \text{rectangle} \end{array} \} \]

\[ X_F = \{ x \in A^{\mathbb{Z}^2} \mid \text{there is no element of } F \text{ in } x \} \]
The Domino Problem on $\mathbb{Z}^2$:

Input: $\mathcal{A}, \mathcal{F}$.

Output: YES if $X_{\mathcal{F}} \neq \emptyset$, NO otherwise.
Theorem (Berger 66, Robinson 71, Mozes 89, Kari 96...):

$DP(\mathbb{Z}^2)$ is undecidable.
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$DP(\mathbb{Z}^2)$ is undecidable.

Reduction to the Halting Problem.
$X_{H,V}:$

$H$ and $V$ SFTs on $\mathbb{Z}$ with alphabet $\mathcal{A}$. 
$X_{H,V}$:

$H$ and $V$ SFTs on $\mathbb{Z}$ with alphabet $\mathcal{A}$. $X_{H,V}$ is the SFT on $\mathbb{Z}^2$ using $\mathcal{A}$ with lines in $H$ and columns in $V$. 
\[ \mathcal{F}(H) = \{ \bullet\bullet, \circ\circ \} \]
\( \mathcal{F}(H) = \{ \bullet \bullet, \circ \circ \} \)
\( \mathcal{F}(V) = \{ \bullet \bullet, \circ \circ \} \)
\( \mathcal{F}(H) = \{ \bullet\bullet, \circ\circ \} \)

\( \mathcal{F}(V) = \{ \bullet\bullet, \circ\circ \} \)

Compatible
\[ \mathcal{F}(H) = \{ \bullet\bullet, \circ\circ \} \]

\[ \mathcal{F}(V) = \{ \circ\circ\circ, \bullet\bullet\bullet, \bullet\bullet \} \]
\( \mathcal{F}(H) = \{ \bullet \bullet , \circ \circ \} \)

\( \mathcal{F}(V) = \{ \circ \circ \circ , \bullet \circ \circ \circ , \bullet \bullet \bullet \} \)

Incompatible
$DP_H$: 

$H$ an SFT on $\mathbb{Z}$. 

$DP_H$: 

$H$ an SFT on $\mathbb{Z}$. 

Input: set of vertical forbidden patterns $\mathcal{F}_V$. 

Remark: The answer depends on $H$!
\[DP_H:\]

\(H\) an SFT on \(\mathbb{Z}\).
Input: set of vertical forbidden patterns \(\mathcal{F}_V\).
Output: \(YES\) if \(X_{H,V} \neq \emptyset\), \(NO\) otherwise.
**DP_H:**

H an SFT on \( \mathbb{Z} \).

Input: set of vertical forbidden patterns \( \mathcal{F}_V \).

Output: YES if \( X_{H,V} \neq \emptyset \), NO otherwise.

**Remark:**

The answer depends on \( H \)!
\[ H = \]
\[ H = \]

\begin{tikzpicture}
  \node[draw, circle, fill=blue!20] (A) at (0,0) {};
  \node[draw, circle, fill=yellow!20] (B) at (1,1) {};
  \node[draw, circle, fill=pink!20] (C) at (1,-1) {};
  \draw[->] (A) to (B);
  \draw[->] (B) to (C);
  \draw[->] (C) to (A);
\end{tikzpicture}
\[ H = \]
$H = \begin{array}{c}
\text{Diagram of a graph with nodes and edges.}
\end{array}$
\[ H = \]

\[ DP_H \] is decidable with this \( H \).
$H =$
\[ H = \]

\[ \begin{array}{ccc}
  & \rightarrow & \\
  \downarrow & & \downarrow \\
  \circ & \rightarrow & \circ \\
  & \rightarrow & \\
  \downarrow & & \downarrow \\
  \circ & \rightarrow & \circ \\
\end{array} \]
$H =$
$H =$
$H =$
\[ H = \]

\[ DP_H \] is decidable with this \( H \).
$H = \ldots$
\( H = \)
$H = \text{DP}$

$H$ is decidable with this.
\( H = \)

\( DP_H \) is decidable with this \( H \).
∀ v, v ⇒ reflexive type
\[ \forall \nu, \nu \Rightarrow \text{reflexive type} \]
\[ \forall \nu, \nu, \nu \Rightarrow \text{symmetric type} \]
\[ \forall v, \quad v \implies \text{reflexive type} \]

\[ \forall v, w, \quad v \lor w \implies \text{symmetric type} \]

\[ \implies \text{state-split cycle type} \]
Question:

These are examples with easy decidability. What about other $H$s?

Theorem (Aubrun-E.-Sablik, 2020):
For any other $H$ with length-2 constraints, $DP_H$ is undecidable.

Idea of the proof:
We reduce to $DP(\mathbb{Z}_2)$. We show that for any $Y$ SFT on $\mathbb{Z}_2$, there are vertical rules $V^Y$ so that $X_H, V^Y$ reproduces the configurations of $Y$. 
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We reduce to... $DP(\mathbb{Z}^2)$.
We show that for any $Y$ SFT on $\mathbb{Z}^2$, there are vertical rules $V_Y$ so that $X_{H,V_Y}$ reproduces the configurations of $Y$. 
Proposition (Aubrun-E.-Sablik, 2020):

For any $Y$ SFT on $\mathbb{Z}^2$, for any “complicated enough” $H$ with length-2 constraints, there is some $V_Y$ so that $X_{H,V_Y}$ reproduces the configurations of $Y$.

Thank you for your attention!
Possible extensions:
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- $H$ with more complex constraints
  
  $\mathcal{A} = \{a, b\}$
  
  $\mathcal{F} = \{aba, bab, bbb\}$
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- $H$ with more complex constraints
  \[ \mathcal{A} = \{a, b\} \]
  \[ \mathcal{F} = \{aba, bab, bbb\} \]

- higher dimension with $H$ on $\mathbb{Z}^k$, input $V$ on $\mathbb{Z}^{d-k}$, $X_{H,V}$ on $\mathbb{Z}^d$. 
\[ H = (\alpha, \Gamma, \gamma) \quad (\beta, \gamma) \quad (b, C^1, c) \]

\[ Y = \{\tau_1, \tau_2, \tau_3\} \]
\[ H = \begin{array}{c}
\gamma \\
\alpha \\
\beta \\
\Gamma
\end{array} \quad \begin{array}{c}
\gamma \\
a \\
b \\
C^1
\end{array} \quad \begin{array}{c}
c
\end{array} \]

\[ Y = \{ \tau_1, \tau_2, \tau_3 \} \]
$H = (\alpha, \Gamma, \gamma, a, C^1, c) \quad Y = \{\tau_1, \tau_2, \tau_3\}$

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>$c$</td>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
<td>$a$</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>$c$</td>
<td>$a$</td>
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<td>$a$</td>
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</tr>
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<td>$c$</td>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
<td>$a$</td>
<td></td>
</tr>
</tbody>
</table>
\[ H = \begin{array}{ccc}
\alpha & \beta & \Gamma \\
\gamma & a & C^1 \\
\end{array} \]

\[ Y = \{ \tau_1, \tau_2, \tau_3 \} \]
$H = \begin{array}{c}
\alpha \\
\Gamma \\
\gamma
\end{array} \begin{array}{c}
\beta \\
\gamma \\
\alpha
\end{array} \begin{array}{c}
b \\
a \\
C^1
\end{array} \begin{array}{c}
c
\end{array}$

$Y = \{\tau_1, \tau_2, \tau_3\}$
\( H = \alpha \Gamma \gamma \)

\[ Y = \{\tau_1, \tau_2, \tau_3\} \]
\[ H = \alpha \Gamma \beta \gamma \alpha \beta \gamma \alpha \]

\[ Y = \{ \tau_1, \tau_2, \tau_3 \} \]
Table of the main cases, each of them illustrated with an example.

<table>
<thead>
<tr>
<th>Loops</th>
<th>No loop</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Diagram 1" /></td>
<td><img src="image2.png" alt="Diagram 2" /></td>
</tr>
<tr>
<td><img src="image3.png" alt="Diagram 3" /></td>
<td><img src="image4.png" alt="Diagram 4" /></td>
</tr>
<tr>
<td><img src="image5.png" alt="Diagram 5" /></td>
<td><img src="image6.png" alt="Diagram 6" /></td>
</tr>
<tr>
<td><img src="image7.png" alt="Diagram 7" /></td>
<td><img src="image8.png" alt="Diagram 8" /></td>
</tr>
<tr>
<td><img src="image9.png" alt="Diagram 9" /></td>
<td><img src="image10.png" alt="Diagram 10" /></td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Case 1.1</th>
<th>Case 1.2</th>
<th>Case 1.3</th>
<th>Case 2.1</th>
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<th>Case 3.1</th>
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