Median in median graphs in linear time

EJCIM 2020

Laurine Bénéteau, Jérémie Chalopin, Victor Chepoi, Yann Vaxès

Aix-Marseille Université, Laboratoire d'Informatique et Systèmes

17 Juin 2020

Preliminaries

Median set

Definition (Distance)

For each $u, v \in V$, $d(u, v)$ is the minimum number of edges in a (u, v)-path.

Median set

Definition (Distance)

For each $u, v \in V$, $d(u, v)$ is the minimum number of edges in a (u, v)-path.

Definition (Median function)

Let $w : V \to \mathbb{R}^+$ be a weight function

$$F_w(x) = \sum_{v \in V} w(v) d(x, v)$$

Median set

Definition (Distance)

For each $u, v \in V$, $d(u, v)$ is the minimum number of edges in a (u, v)-path.

Definition (Median function)

Let $w : V \to \mathbb{R}^+$ be a weight function

$$F_w(x) = \sum_{v \in V} w(v) d(x, v)$$

Definition (Median set)

$$\mathrm{Med}_w(G) = \arg\min F_w$$

Median set

Definition (Distance)

For each $u, v \in V$, $d(u, v)$ is the minimum number of edges in a (u, v)-path.

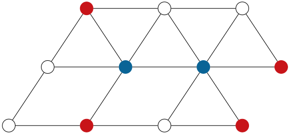
Definition (Median function)

Let $w : V \to \mathbb{R}^+$ be a weight function

$$F_w(x) = \sum_{v \in V} w(v) d(x, v)$$

Definition (Median set)

$$\mathrm{Med}_w(G) = \arg\min F_w$$



Median set

Definition (Distance)

For each $u, v \in V$, $d(u, v)$ is the minimum number of edges in a (u, v)-path.

Definition (Median function)

Let $w : V \to \mathbb{R}^+$ be a weight function

$$F_w(x) = \sum_{v \in V} w(v) d(x, v)$$

Definition (Median set)

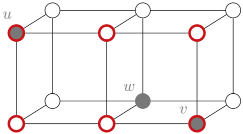
$$\mathrm{Med}_w(G) = \arg\min F_w$$

Goal : Compute $\mathrm{Med}_w(G)$ faster than the distance matrix of G

Median graphs

Definition (Interval)

$$I(u, v) = \{x \in V \ : \ d(u, v) = d(u, x) + d(x, v)\}.$$



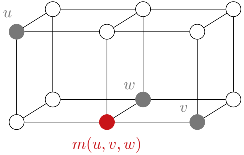
Median graphs

Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u,v,w \in V,\ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$



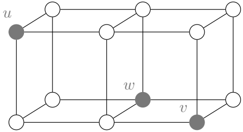
Median graphs

Definition (Interval)

$$I(u,v) = \{x \in V \;:\; d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u, v, w \in V,\ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$



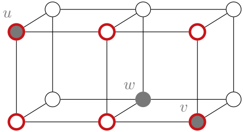
Median graphs

Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u, v, w \in V,\ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$



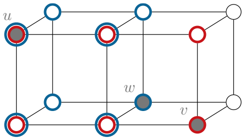
Median graphs

Definition (Interval)

$$I(u, v) = \{x \in V \ : \ d(u, v) = d(u, x) + d(x, v)\}.$$

Definition (Median graph)

$$\forall u, v, w \in V, \ |I(u, v) \cap I(u, w) \cap I(v, w)| = 1$$



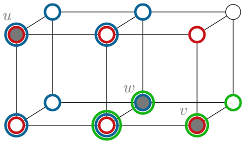
Median graphs

Definition (Interval)

$$I(u, v) = \{x \in V \ : \ d(u, v) = d(u, x) + d(x, v)\}.$$

Definition (Median graph)

$$\forall u, v, w \in V, \ |I(u, v) \cap I(u, w) \cap I(v, w)| = 1$$



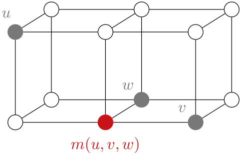
Median graphs

Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u,v,w \in V,\ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$



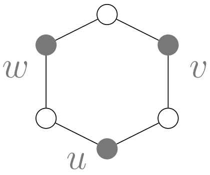
Median graphs

Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u,v,w \in V,\ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$



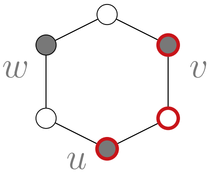
Median graphs

Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u, v, w \in V, \ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$



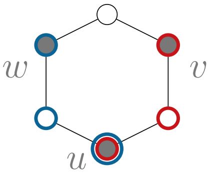
Median graphs

Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u,v,w \in V,\ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$



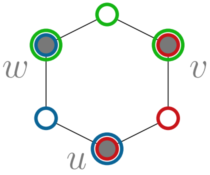
Median graphs

Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u,v,w \in V, \ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$



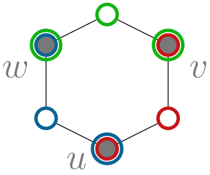
Median graphs

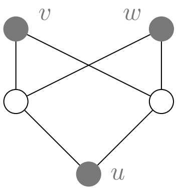
Definition (Interval)

$$I(u,v) = \{x \in V \;:\; d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u,v,w \in V,\ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$





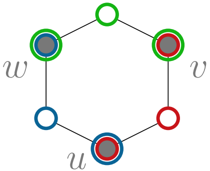
Median graphs

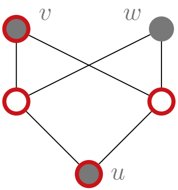
Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u,v,w \in V,\ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$





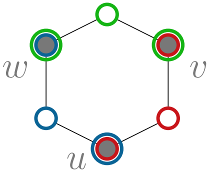
Median graphs

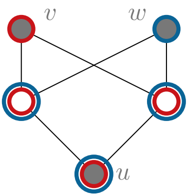
Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u, v, w \in V, \ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$





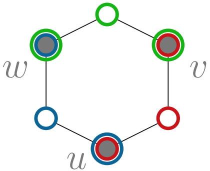
Median graphs

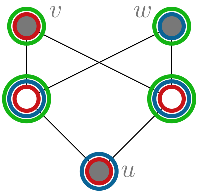
Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u, v, w \in V, \ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$





Median graphs

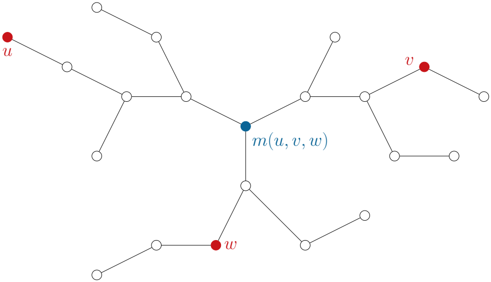
Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u, v, w \in V, \ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$

Trees :



Median graphs

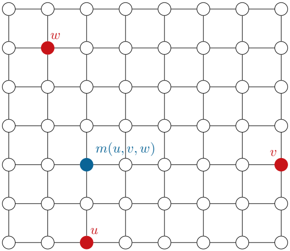
Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u,v,w \in V,\ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$

Grids :



Median graphs

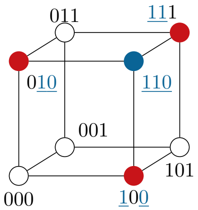
Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u, v, w \in V, \ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$

Hypercubes :



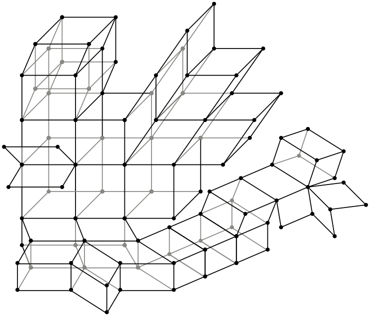
Median graphs

Definition (Interval)

$$I(u,v) = \{x \in V \ : \ d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u,v,w \in V, \ |I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$

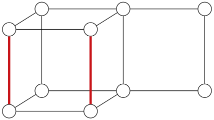


Halfspaces

Θ-classes

Definition (Oppositeness relation Θ_0)

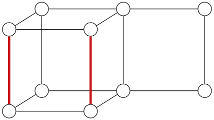
$e\Theta_0 e'$ iff e and e' are two edges on the opposite sides of a square



Θ-classes

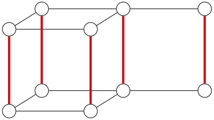
Definition (Oppositeness relation Θ_0)

$e \Theta_0 e'$ iff e and e' are two edges on the opposite sides of a square



Definition (Parallelism relation Θ)

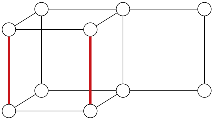
$\Theta = \Theta_0^*$ is the reflexive and transitive closure of Θ_0



Θ-classes

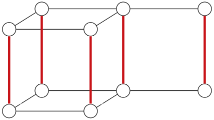
Definition (Oppositeness relation Θ_0)

$e\Theta_0 e'$ iff e and e' are two edges on the opposite sides of a square



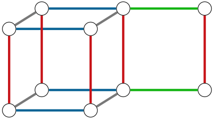
Definition (Parallelism relation Θ)

$\Theta = \Theta_0^*$ is the reflexive and transitive closure of Θ_0



Definition (Θ-classes)

The Θ-classes denotes the equivalence classes of the relation Θ



Computation of the Θ-classes in $O(m)$

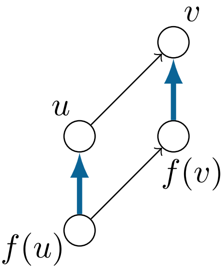
Theorem

The Θ-classes of a median graph with m edges can be computed in $O(m)$ time

Computation of the Θ-classes in $O(m)$

Theorem

The Θ-classes of a median graph with m edges can be computed in $O(m)$ time



Halfspaces (1/2)

Definition (Halfspaces)

Each Θ-classe split G in two convex and gated subgraphs called **halfspaces**

Definition (Convexity)

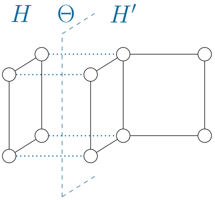
$S \subseteq V$ is convex if

$$\forall u, v \in S, I(u, v) \subseteq S$$

Definition (Gated set)

$S \subseteq V$ gated if

$$\forall x \in V \setminus S,\ \exists x' \in S,\ \forall y \in S,\ x' \in I(x, y)$$



Halfspaces (1/2)

Definition (Halfspaces)

Each Θ-classe split G in two convex and gated subgraphs called **halfspaces**

Definition (Convexity)

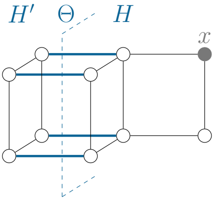
$S \subseteq V$ is convex if

$$\forall u, v \in S, I(u,v) \subseteq S$$

Definition (Gated set)

$S \subseteq V$ gated if

$$\forall x \in V \setminus S,\ \exists x' \in S,\ \forall y \in S,\ x' \in I(x,y)$$



Halfspaces (1/2)

Definition (Halfspaces)

Each Θ-classe split G in two convex and gated subgraphs called **halfspaces**

Definition (Convexity)

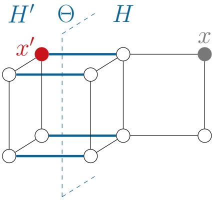
$S \subseteq V$ is convex if

$$\forall u, v \in S, I(u, v) \subseteq S$$

Definition (Gated set)

$S \subseteq V$ gated if

$$\forall x \in V \setminus S,\ \exists x' \in S,\ \forall y \in S,\ x' \in I(x, y)$$



Halfspaces (1/2)

Definition (Halfspaces)

Each Θ-classe split G in two convex and gated subgraphs called **halfspaces**

Definition (Convexity)

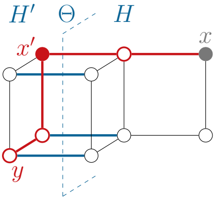
$S \subseteq V$ is convex if

$$\forall u, v \in S, I(u, v) \subseteq S$$

Definition (Gated set)

$S \subseteq V$ gated if

$$\forall x \in V \setminus S,\ \exists x' \in S,\ \forall y \in S,\ x' \in I(x, y)$$

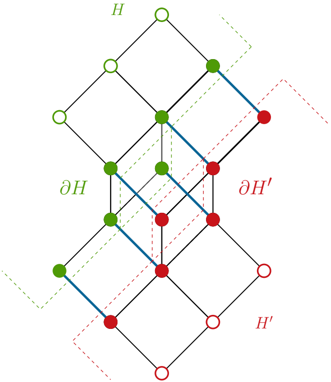


Halfspaces (2/2)

Definition (Boundaries)

The boundary of a halfspace is :

$$\partial H' = \{u \in H' \ : \ \exists v \in H, uv \in \Theta\}$$



Peripheral halfspace

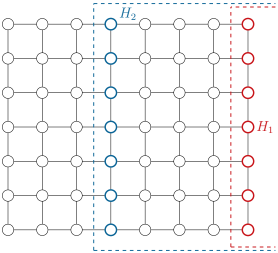
Definition (Peripheral halfspace)

A halfspace H is called peripheral if $H = \partial H$.

Peripheral halfspace

Definition (Peripheral halfspace)

A halfspace H is called peripheral if $H = \partial H$.



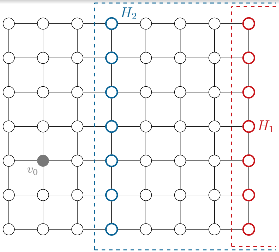
Peripheral halfspace

Definition (Peripheral halfspace)

A halfspace H is called peripheral if $H = \partial H$.

Property

If a halfspace H of G maximizes $d(v_0, H)$ for v_0 in V, then H is a peripheral halfspace

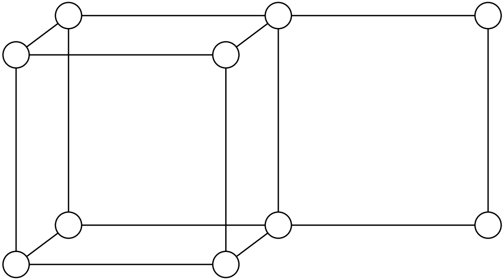


Median set in G

Majority Rule

Property (Majority rule)

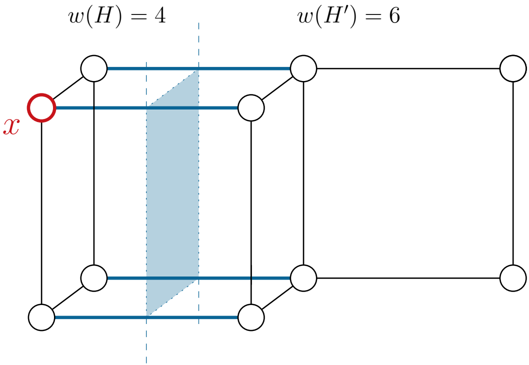
$$\mathrm{Med}_w(G) = \cap\{H \mid w(H) \geq \tfrac{1}{2}w(G)\}$$



Majority Rule

Property (Majority rule)

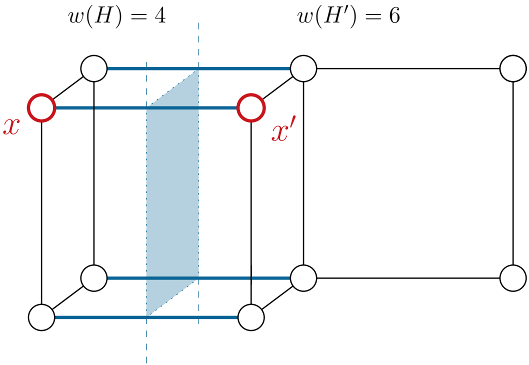
$$\mathrm{Med}_w(G) = \cap\{H \mid w(H) \geq \tfrac{1}{2}w(G)\}$$



Majority Rule

Property (Majority rule)

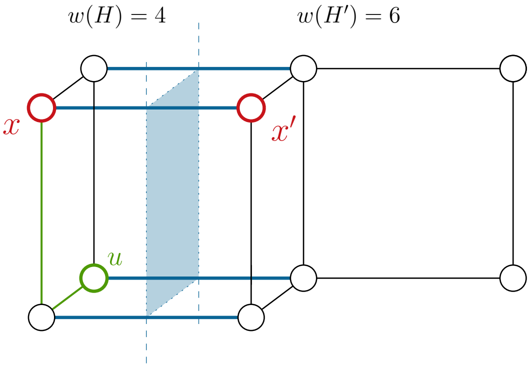
$$\mathrm{Med}_w(G) = \cap\{H \mid w(H) \geq \tfrac{1}{2} w(G)\}$$



Majority Rule

Property (Majority rule)

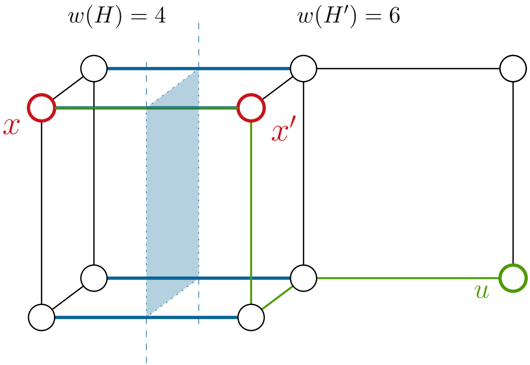
$$\mathrm{Med}_w(G) = \cap\{H \mid w(H) \geq \tfrac{1}{2}w(G)\}$$



Majority Rule

Property (Majority rule)

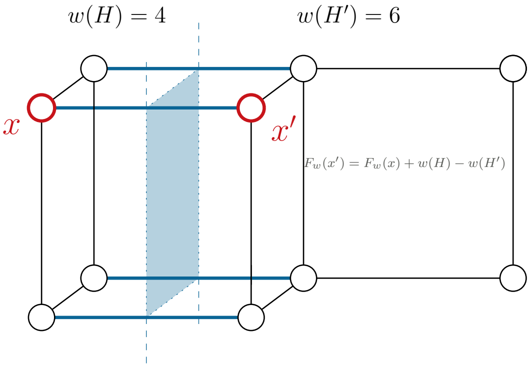
$$\mathrm{Med}_w(G) = \cap\{H \mid w(H) \geq \tfrac{1}{2} w(G)\}$$



Majority Rule

Property (Majority rule)

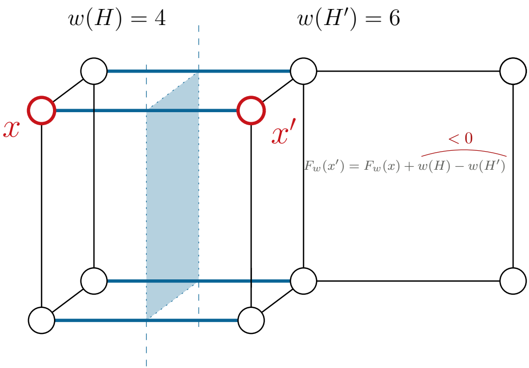
$$\mathrm{Med}_w(G) = \cap\{H \mid w(H) \geq \tfrac{1}{2} w(G)\}$$



Majority Rule

Property (Majority rule)

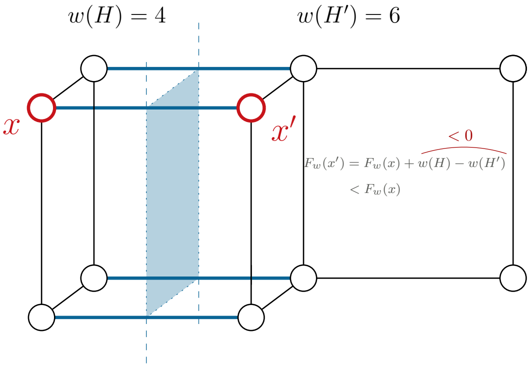
$$\mathrm{Med}_w(G) = \cap\{H \mid w(H) \geq \tfrac{1}{2}w(G)\}$$



Majority Rule

Property (Majority rule)

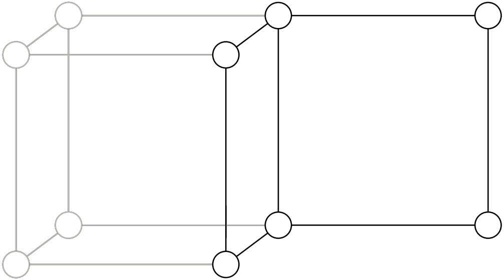
$$\mathrm{Med}_w(G) = \cap\{H \mid w(H) \geq \tfrac{1}{2}w(G)\}$$



Majority Rule

Property (Majority rule)

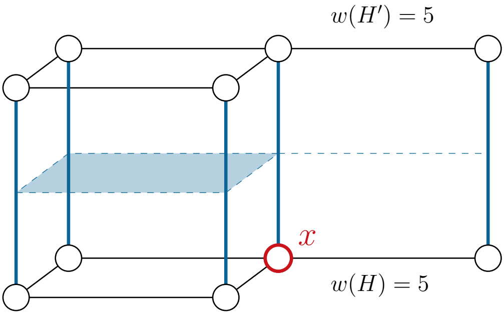
$$\mathrm{Med}_w(G) = \cap\{H \mid w(H) \geq \tfrac{1}{2}w(G)\}$$



Majority Rule

Property (Majority rule)

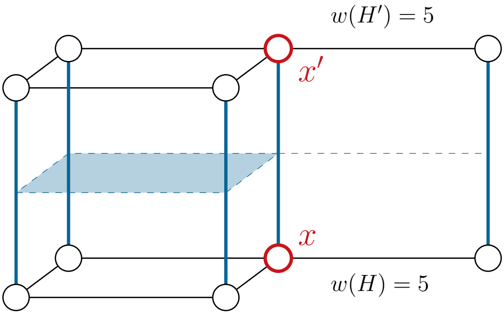
$$\mathrm{Med}_w(G) = \cap\{H \mid w(H) \geq \tfrac{1}{2}w(G)\}$$



Majority Rule

Property (Majority rule)

$$\mathrm{Med}_w(G) = \cap\{H \mid w(H) \geq \tfrac{1}{2}w(G)\}$$



Algorithm

Median $(G = (V, E), \mathbf{\Theta})$:

Order the Θ-classes

For Each Θ-class Θ :

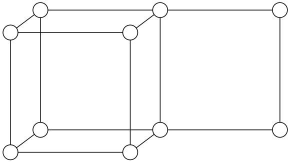
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



Algorithm

Median $(G = (V, E), \mathbf{\Theta})$:

Order the Θ-classes

For Each Θ-class Θ :

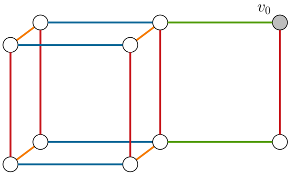
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



Algorithm

Median $(G = (V, E), \mathbf{\Theta})$:

Order the Θ-classes

For Each Θ-class Θ :

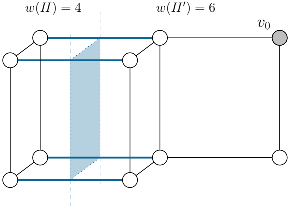
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



Algorithm

Median $(G = (V, E), \mathbf{\Theta})$:

Order the Θ-classes

For Each Θ-class Θ :

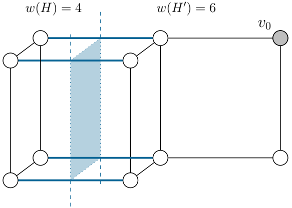
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



Algorithm

Median $(G = (V, E), \mathbf{\Theta})$:

Order the Θ-classes

For Each Θ-class Θ :

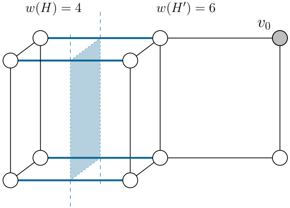
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



Algorithm

Median $(G = (V, E), \mathbf{\Theta})$:

Order the Θ-classes

For Each Θ-class Θ :

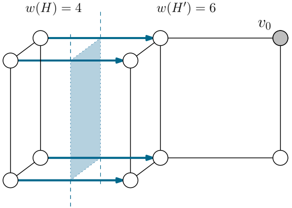
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



Algorithm

Median $(G = (V, E), \boldsymbol{\Theta})$:

Order the Θ-classes

For Each Θ-class Θ :

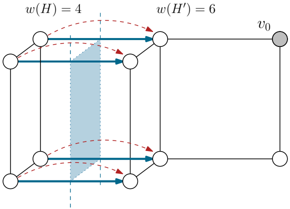
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



Algorithm

Median $(G = (V, E), \mathbf{\Theta})$:

Order the Θ-classes

For Each Θ-class Θ :

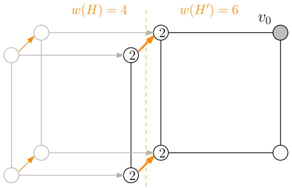
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



Algorithm

Median $(G = (V, E), \mathbf{\Theta})$:

Order the Θ-classes

For Each Θ-class Θ :

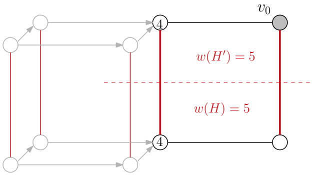
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



Algorithm

Median $(G = (V, E), \mathbf{\Theta})$:

Order the Θ-classes

For Each Θ-class Θ :

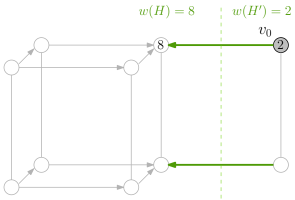
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



Algorithm

Median $(G = (V, E), \mathbf{\Theta})$:

Order the Θ-classes

For Each Θ-class Θ :

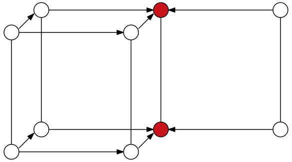
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



Algorithm

Median $(G = (V, E), \mathbf{\Theta})$:

Order the Θ-classes

For Each Θ-class Θ :

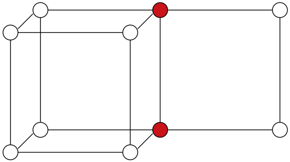
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



Median set computation in $O(m)$ time

Theorem

For each median graph G with m edges and the weighted function w, given the Θ-classes, $\mathrm{Med}_w(G)$ can be computed in optimal linear time $O(m)$

Thank you !